Exponents and Primes: Hints and Solutions

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Please look at the hints only after spending some time working on the problems. Most of the problems on the handout are challenging, especially the last ones, so it is normal to spend more than 1-2 hours on them.

Warm-Up:

1. Use Fermat's Little Theorem.
2. Look at a specific prime p dividing both numbers.
3. Show d is even.
4. You are allowed to use ONLY Fermat's Little Theorem and Lemma 4.
5. Use Lemma 4. For the second note, use Lemmas 4 and 5.
6. Use Fermat's Little Theorem.
7. Use Fermat's Little Theorem and the fact that numbers $2^{2^n}+1, 2^{2^m}+1$ are coprime if $m \neq n$.

First Set of Problems:

1. This is a straightforward application of Lemma 5.
2. Try to look at $(kp+b)^i$ and $(kp^2+b)^i$ modulo p^2 and modulo p^3 by expanding.
3. Look at a_2. What kind of a number is it?
4. The question is destroyed with Lemma 4.
5. Look at the smallest prime divisor of n.
6. Look at a prime dividing n.
7. Look at the smallest prime divisor of n.
8. Prove that 3 divides n. Now look at the second smallest prime divisor of n, and prove it does not exist.
9. Show that one of p, q, r must be even.
10. Let $d = \gcd(m, n)$. Then if $b = a^d$ then $b-1$ and b^r-1, where $n = rd$, have the same prime divisors. Now show that r cannot have an odd prime divisor. Show that if $p|(b+1)$ then $p|b^r-1$.

Second Set of Problems:

1. Use induction on the number of divisors, k. Show that that we can find n with exactly k prime divisors so that $n|(2^n+1)$ and there is a prime dividing 2^n+1 but not dividing n.

2. Look at an odd prime divisor of n. Use corollary 4.

3. What if we look at n^7+2^7? Now work modulo 4. Useful Result 1 may be helpful.

4. Use induction on the number of divisors and something similar to corollary 4, but stronger, since you need 4 new divisors.

5. Show that if $a^{p-1}-1$ is divisible by p^2 then $(p-a)^{p-1}-1$ is not. So if $a^{p-1}-1$ is divisible by p^2 then $(a+1)^{p-1}-1$ is not. Now get a contradiction by looking at specific numbers.

6. The answer is all primes. To rule out the composite numbers is not as hard; to prove it for primes look at a prime p dividing $\dfrac{a^n+1}{a+1}$.

7. The answer is a power of a prime or 12. First deal with odd primes. Look at p, the smallest prime divisor of n and write $n=p^k m$ where $p\nmid m$. Show that $p+m-1$ is corpime with m and so is a power of p. For the highest power of 2 dividing n, use the same idea.

8. Show that $\dfrac{p^p-1}{p-1}$ has a prime factor not congruent to 1 modulo p^2. It is congruent to 1 modulo p from useful result 6. Now show this prime will solve the problem.

9. For a prime p, find n so that $p|(a^n-n)$.

10. Use useful result 7 and consider prime factorization of n.

11. Work modulo a prime.

12. Prove that x is even. Use corollaries 3 and 4.

The solutions to almost all of the problems can be found on Art of Problem Solving.

1 *IMO Problems*,
`http://www.artofproblemsolving.com/Forum/resources.php?c=1&cid=16`

2 *IMO ShortList*,
`http://www.artofproblemsolving.com/Forum/resources.php?c=1&cid=17`

3 *China TSTs*,
`http://www.artofproblemsolving.com/Forum/resources.php?c=37&cid=47`

4 *Russian Olympiads*,
`http://www.artofproblemsolving.com/Forum/resources.php?c=143&cid=61`

5 *USA TSTs*,
`http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=35`