# Strategic Traders and Liquidity Crashes

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**Abstract**

We discuss two papers investigating the effect of strategic traders on market liquidity. Traders anticipate a potential temporary drop in price of an asset and a recovery soon afterwards. They need to decide on their optimal holdings of the asset throughout the period. The drop in price is caused by the fact that traders are worried that they will be forced to sell after the sharp drop in price, but before the recovery. This forces the traders to be more prone to selling before an exogenous shock occurs, which by itself may cause a crash in the absence of any shock. We provide feedback on the papers as well as offer potential extensions. In particular, incorporating transaction costs and increasing the payoffs in case an investor is forced to sell leads to more realistic results.

# 1 Introduction

Liquidity crashes are a very important issue in finance. They are caused by a market imbalance in supply and demand for stocks, whereby there are a lot more sellers than buyers. The excess supply leads to buyers seeking lower prices when buying the stock. Selling even a small position may significantly move prices. Combining this property with the fact that a lot of people are selling at the same time results in large price drops over short periods.

It is uncommon to observe liquidity crashes at an aggregate market level. However, when they do occur, then they have a significant impact on the markets. We recount some notable recent cases of such crashes. On October 19, 1987, the Dow Jones Industrial Average (DJIA) dropped 22.6%, its largest single-day drop ever. It is believed that one of the major causes of this decline was the widespread use of algorithmic trading strategies, which would automatically sell stocks upon hitting specified limits. A similar situation occurred on May 6, 2010, known as the "Flash Crash", during which the DJIA plunged 6% in 5 minutes, and recovered shortly thereafter. On October 28, 1997, the DJIA dropped 7%, and there were serious issues with liquidity, as documented in the SEC Legal Staff Bulletin No. 8 describing the crash.

We discuss two papers investigating how a liquidity crash may arise in presence of strategic traders. The traders anticipate a potential temporary drop in price of an asset and a recovery soon afterwards. They can either sell early, or hold the asset waiting for the price to the recover. However, the latter strategy may not work if an investor is forced to liquidate his position before the recovery. Bernardo and Welch (2004) study the situation in which the investor could be hit by an exogenous liquidity shock, forcing him to sell. They find that if the probability of this shock is high enough, than in equilibrium the investor sells the stock *before* the shock even occurs, because he fears that he will be forced to sell at a low price.

Morris and Shin (2004) consider a slightly different mechanishm of this forced selling phenomenon. Traders have loss limits and get fired if these limits are breached. This gives rise to an endogenous shock, in which a trader's limit is breached because other traders sell, driving down the price. An equilibrium strategy in this setting is to set a threshold based on the expected liquidation value, so that if one's own loss limit is above the threshold, the investor sells the stock, and otherwise holds it. We provide numerical results of this solution and find that traders act as if almost everyone will sell the stock, especially if the price sensitivity to the number of sell orders is low. We find these results too extreme, since traders anticipate unreasonably large price declines. To mitigate this issue, we introduce transaction costs into the model, as well as higher payoffs in the case when a limit is breached. These modifications lead to more realistic outcomes in equilibrium, and more stabilty in the system, whereby less investors sell the stock.

We briefly discuss other extensions to the two papers. We consider a multi-period setting of the Morris and Shin model and show that is it possible to have two subsequent price drops in equilibrium. Furthermore, we note that both models lack the possibility of investors buying the stock after the drop but before the recovery. This possibility is explored in a paper by Brunnermeier and Pedersen (2005) and offers new insights on how strategic investors may behave in a situation where liquidity is low.

# 2 Bernardo and Welch Model

There are three dates 0, 1, and 2 and two assets: a risk-free bond paying no interest and a stock in unit supply with a random payoff traded at date 0 and 1 and liquidated at date 2. There are two types of market participants.

The first type are short-term strategic investors, or *traders*, holding the risky asset at date 0. We assume they are risk-neutral, have total measure 1, and have no individual price impact. They may sell the stock at date 0, or hold it until liquidation at date 2. However, these investors experience a liquidity shock with probability $s$ at date 1, which forces them to sell the stock at the date 1 price. This is undesirable for the investors since the date 1 price is low due to the selling that already occurred at date 0.

The second type are long-term investors, or *market-makers*, who are buying the asset from the short-term traders. We assume the market-makers are risk-averse, and as a result, have decreasing demand in the number of shares offered by the traders.

In this summary of the paper we will only consider the case when the random payoff $Z$ of the stock has a normal distribution with mean $v$ and variance $\sigma^2$. We also assume the market-maker has negative exponential utility $u(W) = -\exp(-\gamma W)$, where $\gamma$ is his risk-aversion coefficient. All of the information is common knowledge among all market participants.

## 2.1 Prices

Let $\alpha$ be the number of shares sold at date 0, and $\beta$ the number solid at date 1. Denote by $p_0(\alpha)$ and $p_1(\alpha, \beta)$ the date 0 and date 1 prices, respectively.

Assume the market-maker has initial wealth $W_0$. If he buys $\alpha$ shares at date 0, and ignores potential future investment opportunities, then his wealth at date 2 is $W_2 = W_0 + \alpha(Z - p_0)$. He then sets the price $p_0(\alpha)$ so that he is indifferent between holding the shares and investing all his wealth in the bond:

$$\begin{aligned} \mathbb{E}(-\exp(-\gamma W_2)) &= \mathbb{E}(-\exp(-\gamma W_0)) \\ \Rightarrow \mathbb{E}(W_0 + \alpha(Z - p_0)) - \gamma \frac{\text{var}(W_0 + \alpha(Z - p_0))}{2} &= W_0 \\ \Rightarrow p_0(\alpha) &= v - c\alpha \end{aligned} \tag{1}$$

where $c = \frac{\gamma\sigma^2}{2}$.

We now derive the date 1 price $p_1$. The market-maker already bought $\alpha$ shares at date 0, and if he buys $\beta$ more, his date 2 wealth is $W_2' = W_2 + \beta(Z - p_1)$. He has to be indifferent between $W_2$ and $W_2'$ so that:

$$\begin{aligned} \mathbb{E}(-\exp(-\gamma W_2')) &= \mathbb{E}(-\exp(-\gamma W_2)) \\ \Rightarrow \mathbb{E}(W_2 + \beta(Z - p_1)) - \gamma \frac{\text{var}(W_2 + \beta(Z - p_1))}{2} &= \mathbb{E}(W_2) - \gamma \frac{\text{var}(W_2)}{2} \\ \Rightarrow p_1 &= v - c(2\alpha + \beta) \end{aligned} \tag{2}$$

## 2.2 Strategic Trader's Problem

Consider a strategic trader who believes that $\alpha$ shares will be sold at date 0. He also believes that liquidity shocks are perfectly correlated; thus if a shock occurs, the remaining $\beta = 1 - \alpha$ shares will be sold. If the trader sells $\alpha$ shares at date 0, his expected payoff is the price $p_0(\alpha)$. If he decides to hold the stock until liquidation, then he expects to receive the date 1 price $p_1(\alpha, 1-\alpha)$ if he experiences the shock (with probability $s$), and otherwise he expects to receive the random liquidation payoff $Z$, with expected value $v$. Therefore it is optimal for the investor to sell at date 0 iff:

$$p_0(\alpha) \geq s \cdot p_1(\alpha, 1-\alpha) + (1-s) \cdot v \tag{3}$$

## 2.3 Equilibrium

We will look for a symmetric equilibrium, in which each trader sells with the same probability $\alpha \in [0,1]$, and this is common knowledge. Consider the expected benefit to sell at date 0 defined based on (3):

$$F(\alpha) = p_0(\alpha) - (s \cdot p_1(\alpha, 1-\alpha) + (1-s) \cdot v) \tag{4}$$

Then selling ($\alpha^* = 1$) is a pure strategy Nash equilibrium iff $F(1) \geq 0$. Holding the stock ($\alpha^* = 0$) is a pure strategy Nash equilibrium iff $F(0) \leq 0$. Finally, there is a mixed strategy Nash equilibrium $\alpha^* \in (0,1)$ iff $F(\alpha^*) = 1$. Substituting the formulas for $p_0(\alpha), p_1(\alpha, 1-\alpha)$ from (1), (2) into (4) we obtain the following characterization of the equilibrium depending on $s$:

**Theorem 2.1.** *If liquidity shocks are perfectly correlated, then there is a unique symmetric Nash equilibrium defined by:*

$$\alpha^* = \begin{cases} \frac{s}{1-s}, & \text{if } s \leq \frac{1}{2} \\ 1, & \text{if } s > \frac{1}{2} \end{cases} \tag{5}$$

*Proof.* See Appendix. □

There are two distinct cases depending on the probability of a liquidity shock. If the probability is low, then only some of the traders sell the stock at date 0. The number of them selling is increasing in $s$ and convex. Thus if a trader believes that more of the other market participants will sell, then he is more prone to selling. The other case is when the shock probability is high. In this situation *all* of the traders sell the stock. The traders have so much fear that they may be forced to liquidate at the low price, then they sell at date 0 before a shock can even occur at date 1.

Bernardo and Welch also consider more advanced settings of this model. They numerically calculate the equilibrium allocation $\alpha^*$ when market-makers have CRRA utility and find it depends on other parameters in the model, such as wealth and risk aversion, and not just $s$. They also look at traders with margin constraints, which gives rise to an endogenous probability of a shock $s(\alpha)$, that depends on the number of traders selling at date 0. In this case multiple equilibria could occur.

We believe this model is nice since it is very tractable and produces the desired result of traders selling in fear of being forced to sell after others have already sold the stock. However, due to the disrceteness of the set-up the price equilibrium appears too extreme (especially if the shock probability is high – in which case everyone sells). Furthermore, all the traders are identical, which is not very realistic.

# 3 Morris and Shin Model

Morris and Shin consider a similar setting to the Bernardo and Welch model, but with heterogenous traders who have loss limits. There are two dates 0 and 1. There is a risk-free bond paying zero interest and a stock traded at date 0. The stock is liquidated at date 1 with a random payoff $Z \sim N(v, \sigma^2)$.

The market-makers are modeled in the exact same way as before. They set a price $p_0(\alpha) = v - c\alpha$ to hold $\alpha$ shares of the stock.

The framework for the traders is a more advanced. They are still risk-neutral, have total measure 1, and no individual price impact. However, their date 0 sell orders are executed at an uncertain price. If a trader submits a sell order, and $\alpha$ orders are submitted in total, then this order is filled at a price $v - cY$, where $Y \sim U[0, \alpha]$. Furthermore, each trader has a loss limit $q_i = \theta + \epsilon_i$ where $\epsilon_i$ are indepdendent $U[-\epsilon, \epsilon]$ is an idiosyncratic component of the limit. If this limit is breached, then the trader is fired and receives a payoff of 0.

## 3.1 Strategic Trader's Problem

For a trader with loss limit $q_i$ define $\hat{\alpha}_i = \frac{v-q_i}{c}$ to be the number of total shares that need to be sold at date 0 in order to breach his limit. If he chooses to hold the stock until liquidation, he receives a different payoff depending on if his limit is breached or not:

$$u(\alpha, q_i) = \begin{cases} v, & \text{if } \alpha \leq \hat{\alpha}_i \\ 0, & \text{if } \alpha > \hat{\alpha}_i \end{cases} \tag{6}$$

If he chooses to sell the stock at date 0, there are two cases. If his limit is lower than the lowest possible price $v - c\alpha$, then his sell order will always be executed without breaching the limit. Otherwise, his limit is not breached only when his sell order is executed at a high price. For this it is necessary to have $Y \leq \hat{\alpha}_i$, which occurs with probability $\frac{\hat{\alpha}_i}{\alpha}$. Taking conditional expectations we can determine his payoffs:

$$w(\alpha, q_i) = \begin{cases} v - \frac{1}{2}c\alpha, & \text{if } \alpha \leq \hat{\alpha}_i \\ \frac{\hat{\alpha}_i}{\alpha}(v - \frac{1}{2}c\hat{\alpha}_i), & \text{if } \alpha > \hat{\alpha}_i \end{cases} \tag{7}$$

## 3.2 Equilibrium

We derive a "threshold" equilibrium, in which a trader sets a limit threshold $q^*(v)$ depending on the expected liquidation value $v$, so that he sells the stock if his own loss limit is higher than the threshold, and holds it otherwise.

**Theorem 3.1.** *There is a unique threshold equilibrium, in which $q^*$ is the solution to:*

$$v - q_i = c \exp\left(\frac{q_i - v}{2(q_i + v)}\right) \tag{8}$$

*Proof.* In equilibrium a trader with loss limit equal to $q^*$ believes that $\alpha \sim U[0,1]$. As a result, we have:

$$\int_0^1 u(\alpha, q^*) - w(\alpha, q^*) d\alpha = 0 \tag{9}$$

Substituting the formulas for $u$ and $w$ from (6), (7) the result follows. A more detailed proof is provided in the Appendix. □

Figure 1 in the Appendix shows the dependence of the optimal loss limit threshold $q^*$ on the price sensitivity $c$. The threshold is decreasing in $c$, so that more traders sell if the price reacts more aggressively to the sell orders. Furthermore, the threshold always lies above the lowest possible price $v - c$ (if everyone sells) and below the expected liquidation value $v$. This makes sense since selling with a loss limit below $v - c$ is a dominated strategy – since the limit will never be breached, while holding when the limit is above $v$ is a dominant strategy – since the limit will always be breached.

It is important to note from the figure that the threshold is close to the lowest possible price, and very close when the price sensitivity is small. Thus the traders act as if almost all other traders will sell the stock. Furthermore they end up selling even if their loss limit is significantly below $v$. Larger price sensitivity leads to a greater difference between the threshold and the lowest price, because traders with very low loss limits are unlikely to sell the stock, and other traders realize that.

We believe that while the equilibrium produces the nice effect of traders aggressively selling in fear of their loss limits getting breached, the optimal limit threshold appears too extreme. In particular, it is very uncommon in real life for *any* short-term trader to have a loss limit of 20% or more below the price, while in figure 1 we are getting loss limits as low as 70% below the price. Also, it is unrealistic for a trader to have the very low payoff of 0 if his limit is breached. Usually, he would still be able to liquidate the stock, just at a much lower price – along the lines of the Bernardo and Welch model.

To address these issues, we propose two minor modifications to the model. First, if a trader's loss limit is breached, his payoff is $R > 0$. Second, we introduce transaction costs $\tau$ as a percentage of executed price for traders who sell at date 0. Using the same method as in the paper we are able to derive the equilibrium condition.

**Theorem 3.2.** *With $R > 0$ is not too high, and transaction costs $\tau \in [0, 1)$, there is a unique threshold equilibrium, in which $q^*$ is the solution to:*

$$v - q_i = c \exp\left(\frac{(q_i - v)(1 - \tau) - 4v\tau}{2(v + q_i)(1 - \tau) - 4R}\right) \tag{10}$$

*Proof.* See Appendix. □

Figure 2 in the Appendix shows the dependence of the optimal loss limit threshold $q^*$ on the price sensitivity $c$, for different settings of the parameters $\tau$ and $R$. We see that with no transaction costs and a payoff $R$ of 0, the loss limit threshold is very close to the lowest possible price when $c \leq 20$. Slightly higher transaction costs ($\tau = 5\%$) and payoff ($R = 20, 40$) increases the threshold. If we further increase the transaction costs to 20%, the threshold significantly rises and is around half-way between the lowest possible price and the expected liquidation value. We believe that this is a more realistic scenario – since now the optimal loss limit is quite close to the liquidation value. It is also important to note that a higher limit threshold means that less traders are selling and the drop in price is not as sharp. Thus higher transaction costs and a larger payoff in the case of a limit breach leads to better stability in the system – which is important from a regulatory perspective, if we are trying to prevent, or at least mitigate, liquidity crashes.

# 4 Conclusion and Further Extensions

We have discussed two papers dealing with liquidity crashes. In the Bernardo and Welch model, traders may be forced to sell a stock at a low price due to an external liquidity shock. When the probability of a shock is high enough, all traders sell the stock before the shock can even occur, leading to a crash.

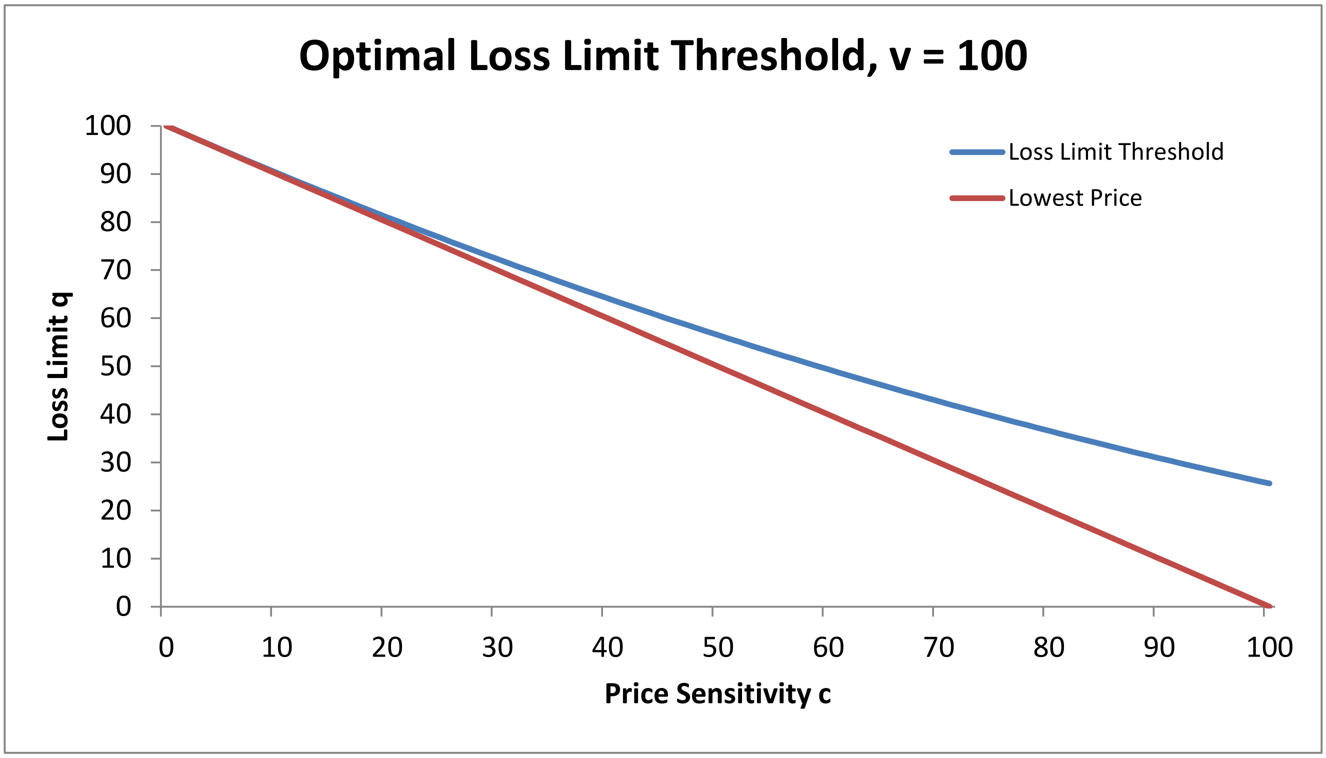
The model by Morris and Shin allows for heterogenous traders with different loss limits. If these limits are breached, the traders get fired. This causes the traders to pre-emptively sell since they are worried that enough other traders will sell, thus pushing the price below their loss limit. In fact, if the price sensitivity is low, traders act as if almost everyone will sell. We find these results a bit too extreme, and to fix this issue, introduce transaction costs as well as a higher payoff in case the limit is breached. This leads to a more realistic equilibrium, whereby investors don't sell so aggressively. Furthermore, in this situation the system is more stable, since the drop in price is not as large.

We outline some more extensions to these papers. It would be nice to have a multi-period model with heterogenous investors leading to liqudity crashes. In real life, while liquduity crashes tend occur over short periods of time, they still involve several subsequent price drops. In the Appendix we provide an extension of the Morris and Shin model where in equilibrium there are two price drops. The idea is to have two types of strategic traders – the first (e.g. a hedge fund) who can trade at dates 1 and 2, and the second (e.g. a pension fund) who reacts slower and as a result can only trade at date 2. Then the first type will sell at date 1 (anticipating the further drop at date 2), while the second will sell at date 2.

Neither paper explores the possibility of investors *buying* at the low price right after the crash, but before the recovery. This investment strategy offers a very favorable return as long as the trader is able to buy low and hold on to the stock. Brunnermeier and Pedersen (2005) study this situation in detail. Their model involves a large investor who is forced to unwind his position in a stock. Other traders recognize that and on purpose sell *with him*, driving the price even lower than what it would have been had the investor sold alone. The traders start buying right at the point when the selling is finished, and earn significant profits as the price recovers due to the buying. Their model leads to an equilibrium, in which the strategic traders further exacerbate the illiquidity in the market because they recognize that the market is illiquid and can take advantage of it.

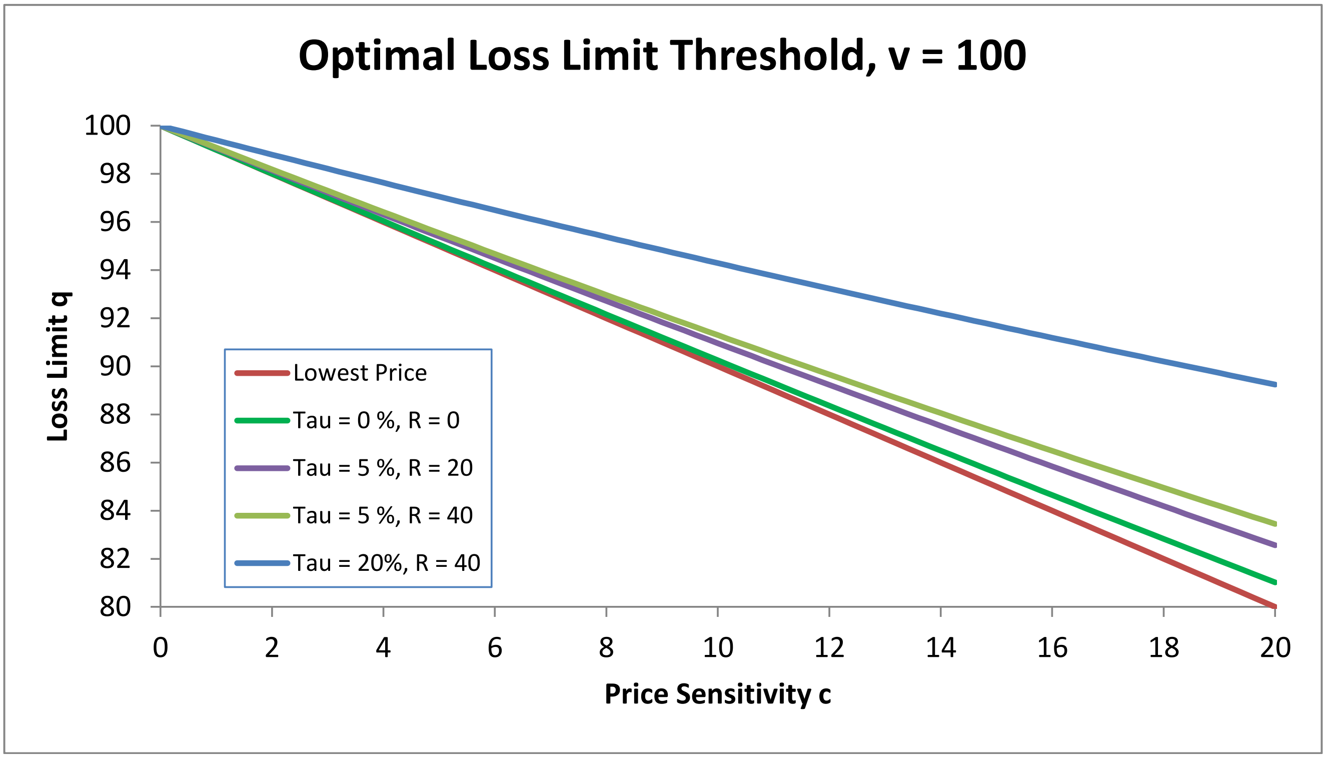
# 5 Appendix

Figure 1: Optimal Loss Limit Threshold




This figure shows the dependence of the optimal loss limit threshold on price sensitivity. The expected liquidation price $v$ is set to 100. The red curve represents the lowest possible price $v - c$, while the blue one is the threshold. The threshold is decreasing in $c$, so that more traders sell if the price reacts more aggressively to the sell orders. Furthermore, the threshold always lies above the lowest possible price $v - c$ (if everyone sells) and below the expected liquidation value $v$.

Figure 2: Optimal Loss Limit Threshold, with Transaction Costs and Higher Payoffs




This figure shows the dependence of the optimal loss limit threshold on price sensitivity. The expected liquidation price $v$ is set to 100. The red curve represents the lowest possible price $v - c$. We vary the transaction costs $\tau$ and the payoff $R$ when the limit is breached. It is evident that higher $\tau$ and $R$ lead to higher optimal loss limit thresholds.

## 5.1 Proof of theorem 5.1

**Theorem 5.1.** *If liquidity shocks are perfectly correlated, then there is a unique symmetric Nash equilibrium defined by:*

$$\alpha^* = \begin{cases} \frac{s}{1-s}, & \textit{if } s \leq \frac{1}{2} \\ 1, & \textit{if } s > \frac{1}{2} \end{cases} \tag{11}$$

*Proof.* Substituting (1), (2) into (4) we get:

$$F(\alpha) = v - c\alpha - (s \cdot (v - c(1+\alpha)) + (1-s) \cdot v) = c(s(1+\alpha) - \alpha) \tag{12}$$

We see that $F(0) > 0$ for all $s > 0$ and thus $\alpha^*$ is never an equilibrium. If $s < \frac{1}{2}$, then $F(\alpha) = 0$ has exactly one solution $\alpha^* = \frac{s}{1-s}$; and furthermore, $F(1) < 0$ so we have a unique Nash equilibrium $\alpha^* = \frac{s}{1-s}$. If $s = \frac{1}{2}$, then $F(\alpha) = 0$ gives $\alpha^* = 1$, while $F(1) = 0$ so we again have a unique Nash equilibrium $\alpha^* = \frac{s}{1-s} = 1$. Finally, if $s > \frac{1}{2}$, then $F(\alpha) = 0$ has no solutions in $[0, 1]$, while $F(1) > 0$, hence there is a unique Nash equilibrium $\alpha^* = 1$. □

## 5.2 Proof of theorem 3.1

**Theorem 5.2.** *There is a unique threshold equilibrium, in which* $q^*$ *is the solution to:*

$$v - q_i = c \exp\left(\frac{q_i - v}{2(q_i + v)}\right) \tag{13}$$

*Proof.* Suppose a trader has loss limit equal to $q^*$. We derive his belief about the distribution $F(z|q^*)$ of $\alpha$, conditional on his loss limit $q^*$. The trader knows that the other traders' sell orders are uinformly distributed in the interval $[\theta - \epsilon, \theta + \epsilon]$. He also knows the other traders follow the threshold strategy, and hence will sell if and only if their own limit $q$ is higher than $q^*$, which occurs with probability:

$$\alpha = \frac{\theta + \epsilon - q^*}{2\epsilon} \tag{14}$$

Therefore $\alpha \leq z$ iff the common signal $\theta$ is at most $\theta^*$, where:

$$\theta^* = q^* - \epsilon + 2\epsilon z \tag{15}$$

However, conditional on the trader's own loss limit being $q^*$, $\theta$ is also distributed uniformly over $[q^* - \epsilon, q^* + \epsilon]$. It follows that $\theta \leq \theta^*$ occurs with probability:

$$\frac{\theta^* - (q^* - \epsilon)}{2\epsilon} = \frac{q^* - \epsilon + 2\epsilon z - (q^* - \epsilon)}{2\epsilon} = z \tag{16}$$

Therefore $F(z|q^*) = z$. In equilibrium the trader with loss limit $q^*$ must be indifferent between selling and holding the asset, therefore:

$$\int_0^1 f(\alpha|v, q^*)[u(\alpha, q^*) - w(\alpha, q^*)]d\alpha = 0 \tag{17}$$

Since $F(z|q^*) = z$, this expression simplifies to:

$$\int_0^1 u(\alpha, q^*) - w(\alpha, q^*)d\alpha = 0 \tag{18}$$

Substituting the formulas for $u$ and $w$ from (6), (7) and using $\hat{\alpha} = \frac{v-q^*}{c}$ this becomes:

$$\frac{1}{2}c \int_0^{\frac{v-q^*}{c}} \alpha d\alpha = \frac{(v - q^*)(v + q^*)}{2c} \int_{\frac{v-q^*}{c}}^1 \frac{1}{\alpha} d\alpha \tag{19}$$

Simplifying, we get:

$$v - q^* = c \exp\left(\frac{q^* - v}{2(q^* + v)}\right) \tag{20}$$

To finish the proof we also need to prove that a trader with loss limit $q_i$ less than $q^*$ would choose to hold the stock, and would sell it if $q_i \geq q^*$. This is rather straightforward by looking at the trader's belief about the conditional distribution $F(z|q)$ of $\alpha$, and comparing the left side of (17) to 0. □

### 5.3 Proof of theorem 5.3

**Theorem 5.3.** *With $R > 0$ is not too high, and transaction costs $\tau \in [0, 1)$, there is a unique threshold equilibrium, in which $q^*$ is the solution to:*

$$v - q_i = c \exp\Big(\frac{(q_i - v)(1-\tau) - 4v\tau}{2(v+q_i)(1-\tau) - 4R}\Big) \tag{21}$$

*Proof.* The proof is similar to that of Theorem 3.1. Equation (17) still holds in equilibrium. However, the payoffs are now different. If the trader decides to hold the stock, his payoff is:

$$u(\alpha, q_i) = \begin{cases} v, \text{ if } \alpha \le \hat{\alpha}_i \\ R, \text{ if } \alpha > \hat{\alpha}_i \end{cases} \tag{22}$$

since his payoff is now $R$ if the limit is breached. If he decides to sell it, his payoff is:

$$w(\alpha, q_i) = \begin{cases} (1-\tau)(v - \frac{1}{2}c\alpha), \text{ if } \alpha \le \hat{\alpha}_i \\ \frac{\hat{\alpha}_i}{\alpha}(1-\tau)(v - \frac{1}{2}c\hat{\alpha}_i) + \frac{\alpha - \hat{\alpha}_i}{\alpha}R, \text{ if } \alpha > \hat{\alpha}_i \end{cases} \tag{23}$$

because of the transaction costs $\tau$. Substituting (22) and (23) into (17) we get:

$$\tau \int_0^{\frac{v-q^*}{c}} v d\alpha + (1-\tau)\frac{1}{2}c \int_0^{\frac{v-q^*}{c}} \alpha d\alpha = \frac{v - q^*}{c}((1-\tau)\frac{v+q^*}{2} - R) \int_{\frac{v-q^*}{c}}^1 \frac{1}{\alpha} d\alpha \tag{24}$$

Simplifying, we get:

$$v - q^* = c \exp\Big(\frac{(q^* - v)(1-\tau) - 4v\tau}{2(v+q^*)(1-\tau) - 4R}\Big) \tag{25}$$

We would again need to show that it is optimal to sell if a trader's limit $q$ is greater than $q^*$, and to hold otherwise. This is straightforward.

Note that we need to have $(1-\tau)\frac{v+q^*}{2} - R > 0$ in order for (24) to have a solution, since the left side of that equation is always non-negative. Thus $R$ cannot be too high. (If it is, then nobody would sell in equilibrium.) □

# 6 Multi-Period Extension of Morris and Shin Model

We extend the Morris and Shin Model to three dates 0, 1, 2. The stock can now be traded at dates 0 and 1, and is liquidated at date 2. There are two types of strategic traders. The first type can trade at both dates 0 and 1, while the second can only trade at date 1. Each type 1 trader has a loss limit $q_i = \theta_1 + \epsilon_i$ with $\epsilon_i$ independent $U[-\epsilon_1, \epsilon_1]$, while each type 2 trader has a loss limit $r_i = \theta_2 + \delta_i$ with $\delta_i$ independent $U[-\epsilon_2, \epsilon_2]$.

In equilibrium the date 1 price is less than or equal to than the date 0 price. Therefore, any type 1 trader, who sells in equilibrium, will do so at date 0. Denote by $\alpha$ the number of type 1 traders selling at date 0, and $\beta$ the number of type 2 traders selling at date 1. Using (1), (2) we derive the prices. The sell orders at date 0 are executed at a random price $v - cY_1$ with $Y_1 \sim U[0, \alpha]$, while date 1 orders are executed at a random price $v - 2c\alpha - cY_2$ with $Y_2 \sim U[0, \beta]$.

We solve for an equilibrium in which both type 1 and type 2 traders use loss limits thresholds $q^*$ and $r^*$, respectively. Type 2 traders, who at date 1 make the deicision to sell or hold the stock, have the following payoffs. If they choose to hold, their payoff is:

$$u_2(\alpha, \beta, r_i) = \begin{cases} v, & \text{if } \beta \leq \hat{\beta}_i \\ 0, & \text{if } \beta > \hat{\beta}_i \end{cases} \tag{26}$$

where $\hat{\beta}_i = \frac{v-2c\alpha-r_i}{c}$. If they choose to sell, their payoff is:

$$w_2(\alpha, \beta, r_i) = \begin{cases} p(\alpha) - \frac{1}{2}c\beta, & \text{if } \beta \leq \hat{\beta}_i \\ \frac{\hat{\beta}_i}{\beta}(p(\alpha) - \frac{1}{2}c\hat{\beta}_i), & \text{if } \beta > \hat{\beta}_i \end{cases} \tag{27}$$

where $p(\alpha) = v - 2c\alpha$. In equilibrium we again have (17) holding, so that:

$$\int_0^1 (u_2(\alpha, \beta, r^*) - w_2(\alpha, \beta, r^*))d\beta = 0 \tag{28}$$

Substituting (26) and (27) and simplifying we get:

$$v - r^* = c \exp\left(\frac{r^* - v - 8\alpha}{2(r^* + v) - 8\alpha}\right) \tag{29}$$

We now look at type 1 traders. If they choose to hold, their payoff is:

$$u_1(\alpha, \beta, q_i) = \begin{cases} v, & \text{if } 2\alpha + \beta \leq \hat{\alpha}_i \\ 0, & \text{if } 2\alpha + \beta > \hat{\alpha}_i \end{cases} \tag{30}$$

where $\hat{\alpha}_i = \frac{v-q_i}{c}$. If they choose to sell, their payoff is:

$$w_1(\alpha, \beta, q_i) = \begin{cases} v - \frac{1}{2}c\alpha, & \text{if } \alpha \leq \hat{\alpha}_i \\ \frac{\hat{\alpha}_i}{\alpha}(v - \frac{1}{2}c\hat{\alpha}_i), & \text{if } \alpha > \hat{\alpha}_i \end{cases} \tag{31}$$

Equation (17) must hold for these traders as well, therefore:

$$\int_0^1 (u_1(\alpha, \beta, q^*) - w_1(\alpha, \beta, q^*))d\alpha = 0 \tag{32}$$

Using (30), (31) we can simplify this expression as well. Combining with (32) we can numerically solve for the equilibrium values of $r^*$ and $q^*$. We have not actually done this, but we believe that the numerical solution should lead to $\alpha > 0, \beta > 0$, which means that in equilibrium the price would drop both at date 0 and at date 1, as desired.

# References


[1] Antonio E Bernardo and Ivo Welch. Liquidity and financial market runs. *The Quarterly Journal of Economics*, 119(1):135–158, 2004.

[2] Markus K Brunnermeier and Lasse Heje Pedersen. Predatory trading. *The Journal of Finance*, 60(4):1825–1863, 2005.

[3] Stephen Morris and Hyun Song Shin. Liquidity black holes. *Review of Finance*, 8(1):1–18, 2004.

[4] SEC Division of Market Regulation Staff. Sec legal staff bulletin no. 8. 1998.