Mathematical Induction

Mathematical Induction is a powerful method for solving problems. Suppose we want to prove a result *for all positive integers n*. To do this, we need to demonstrate two steps:
Base Step: The result holds for $n = 1$.
Induction Step: If the result holds for some positive integer n, then it holds for $n+1$ as well.

For example, let us use mathematical induction to show that for all positive integers n:

$$1 + 3 + \ldots + (2n - 1) = n^2$$

Base Step: Suppose $n = 1$. Then the left side (LS) is $1 + \ldots + 1 = 1$, while the right side (RS) is $1^2 = 1$. Therefore LS = RS, and the base step is complete.
Induction Step: Suppose the result holds for some positive integer n, so that:

$$1 + 3 + \ldots + (2n - 1) = n^2 \tag{1}$$

We need to show the result holds for $n + 1$. We have:

$$\begin{aligned} 1 + 3 + \ldots + (2(n + 1) - 1) &= (1 + 2 + \ldots + (2n - 1)) + (2(n + 1) - 1) \\ &= n^2 + (2(n + 1) - 1) \qquad \text{using equation (1)} \\ &= n^2 + 2n + 1 = (n + 1)^2 \end{aligned}$$

Therefore, the result holds for $n + 1$ as well, and the induction step is complete.
Hence by mathematical induction the result must hold for all positive integers n.

Here is another example. Supppose n people gather at a meeting, and every two of them shake hands. How many handshakes take place?

Let us consider small cases and try to find a pattern. When we have one person, there are obviously 0 handshakes. When person 2 comes along, he shakes hands with the first person, so there are 1 handshakes total. On top of this, person 3 will shake hands with the first two people, so with three people, there are 1 + 2 handshakes total. Person 4 will shake hands with the first three people, so with four people, there are 1 + 2 + 3 handshakes total. You can see the pattern now:

When there are n people, $1 + 2 + \ldots + n - 1$ *handshakes take place.*

We have sort of demonstrated this result using small cases, but to prove the general result, we will use induction.
Base Step: Suppose $n = 1$. Then 0 handshakes take place. Since $1 + \ldots + (1 - 1) = 0$, the result holds for $n = 1$ and the base step is complete.
Induction Step: Suppose the result holds for some positive integer n, and we want to show that it holds for $n+1$ as well. Number the people from 1 to n. The order of the handshakes

doesn't matter, so let us assume people $1, 2, \ldots, n$ shake hands with each other before person $n+1$ shakes hands with anybody. Since the result holds for n, then during this "first stage" $1 + 2 + \ldots + n - 1$ handshakes occur. Now, the only handshakes that are left are the ones involving person $n+1$. There must be n of them (since the person $n+1$ shakes hands with people $1, 2, \ldots, n$). Hence n more handshakes occur during this "second stage". Hence in total the number of handshakes is:

$$(1 + 2 + \ldots + (n-1)) + n = 1 + 2 + \ldots + n$$

Therefore, the result holds for $n+1$ as well, and the induction step is complete.

Problems

1. Prove that for all positive integers n:

   (a) $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$

   (b) $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

   (c) $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \ldots + \dfrac{1}{n \cdot (n+1)} = 1 - \dfrac{1}{n+1}$

2. A sequence is defined by $a_1 = 1$, and for $n \geq 1$, $a_{n+1} = 2a_n + 1$. What is a_{300}?

3. (a) Show that for all positive integers n, the number 5^n has 5 as its units digit.
   (b) Show that for all integers $n \geq 2$, the number 5^n ends in 25.
   (*Note:* The base step should start at $n = 2$, not $n = 1$).

4. Prove that for all integers $n \geq 4$, we have $n! > 2^n$.

5. Show that for all positive inetegers n, $8^n - 1$ is divisible by 7.

6. Assume $x > -1$. For all positive integers n prove that $(1 + x)^n \geq 1 + nx$.

Some Harder Problems

7. Show that for all positive integers n: $\dfrac{1 \cdot 3 \cdot 5 \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \ldots \cdot 2n} \leq \dfrac{1}{\sqrt{2n+1}}$

8. A sequence is defined by $a_1 = 1, a_2 = 2$, and for $n \geq 2$, $a_{n+1} = a_n + 2a_{n-1}$. What is the value of a_{300}?
   Note: To solve this problem, you will need to use *strong induction*. This is the same as regular induction, except in the induction step, you assume that the result holds for all integers less than or equal to n, and not just for the integer n.
   So in this problem, once you figure out a formula for a_n, to show it holds for a_{n+1} in the induction step, you would assume that it holds for *both* a_n and a_{n-1}.

9. Bob comes into a grocery store and wants to buy a calculator for n dollars. He has an infinite number of two-dollar coins and an infinite number of five-dollar bills. Show that for every positive integer $n \geq 4$, he will be able to pay for the calculator without requiring any change.

10. Prove that for every positive integer n, $3^{(2^n)} - 1$ is divisible by 2^{n+2} but not 2^{n+3}.

11. Stacy draws n lines on a sheet of paper, so that no two of them intersect. As a result the lines split up the sheet into different regions. How many such regions are there?

12. Let x be a real number, such that $x + \dfrac{1}{x}$ is an integer. Prove that for every positive integer n, $x^n + \dfrac{1}{x^n}$ is an integer.

Hard Problems

13. (COMC 2013) Alphonse and Beryl play the following game. Two positive integers m and n are written on the board. On each turn, a player selects one of the numbers on the board, erases it, and writes in its place any (positive) divisor of this number as long as it is different from any of the numbers previously written on the board. For example, if 10 and 17 are written on the board, a player can erase 10 and write 2 in its place (as long as 2 has not appeared on the board before). The player who cannot make a move loses. Alphonse goes first.

    (a) Suppose $m = 2^{40}$ and $n = 3^{51}$. Who has the winning strategy, and why?

    (b) Suppose $m = 2^{40}$ and $n = 2^{51}$. Who has the winning strategy, and why?

14. Ler p be a prime number. Prove that if n is a positive integer not divisible by p, then n^{p-1} gives a remainder of 1 when divided by p. (This result is known as *Fermat's Little Theorem* and is very useful in olympiad number theory problems).
    Hint: Try to prove that n^p gives a remainder of n when divided by p.

15. Let $n \geq 4$ be positive integer. Prove that for any positive real numbers $x_1, \ldots, x_n$ the following inequality holds:

$$\frac{x_1}{x_n + x_2} + \frac{x_2}{x_1 + x_3} + \ldots + \frac{x_n}{x_{n-1} + x_1} \geq 2$$