Mathematical Induction Solutions

Problems

1.(a) Base Step: $1 = \frac{1(1+1)}{2}$, so the base case is finished.
Induction Step: Suppose the result holds for some positive integer n. We need to show the result holds for $n+1$. We have:

$$\begin{aligned} 1+2+\ldots+n+1 &= (1+2+\ldots+n)+(n+1) \\ &= \frac{n(n+1)}{2}+(n+1) \qquad \text{since the result holds for } n \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

hence the result holds for $n+1$ as well, and the induction step is complete.

1.(b) Base Step: $1^2 = \frac{1(1+1)(2\cdot 1+1)}{6}$, so the base case is finished.
Induction Step: Suppose the result holds for some positive integer n. We need to show the result holds for $n+1$. We have:

$$\begin{aligned} 1^2+2^2+\ldots+(n+1)^2 &= (1^2+2^2+\ldots+n^2)+(n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6}+(n+1)^2 \qquad \text{since the result holds for } n \\ &= \frac{n+1}{6}(n(2n+1)+6(n+1)) = \frac{n+1}{6}(2n^2+7n+6) \\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \end{aligned}$$

hence the result holds for $n+1$ as well, and the induction step is complete.

1.(c) Base Step: $\frac{1}{1\cdot 2} = 1 - \frac{1}{1+1}$, so the base case is finished.
Induction Step: Suppose the result holds for some positive integer n. We need to show the result holds for $n+1$. We have:

$$\begin{aligned} &\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{(n+1)\cdot(n+2)} = \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}\right)+\frac{1}{(n+1)\cdot(n+2)} \\ &= \left(1-\frac{1}{n+1}\right)+\frac{1}{(n+1)\cdot(n+2)} \qquad \text{since the result holds for } n \\ &= 1-\frac{n+2-1}{(n+1)\cdot(n+2)} = 1-\frac{1}{n+2} \end{aligned}$$

hence the result holds for $n+2$ as well, and the induction step is complete.

2. The first few terms in the sequence are $1, 3, 7, 15, 31, \ldots$. We see that adding 1 to each of these gives us just powers of 2. We claim that the nth term in the sequence has the form $2^n - 1$. Let us prove this claim using induction.
   Base Step: The first term is 1, while $2^1 - 1 = 1$. So the base case is done.
   Induction Step: Assume the result is true for some positive integer n, so that $a_n = 2^n - 1$. For $n + 1$, we have:
   $$a_{n+1} = 2a_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$$
   so the induction step is complete.
   We have thus managed to prove that $a_n = 2^n - 1$, so $a_{300} = 2^{300} - 1$.

3.(a) Base Step: $5^1 = 5$ which clearly ends in 5, so the base step is done.
   Induction Step: Assume the result is true for some positive integer n, so that 5^n ends in 5. Another way to write this is $5^n = 10x + 5$ for some ineteger n. We want to show the result for $n + 1$. We have:
   $$5^{n+1} = 5 \cdot 5^n = 5 \cdot (10x + 5) = 50x + 25 = 50x + 20 + 5 = 10(5x + 2) + 5$$
   which ends in 5, so the induction step is finished.

3.(b) Base Step: (*for $n = 2$*) $5^2 = 25$ which clearly ends in 25, so the base step is done.
   Induction Step: Assume the result is true for some positive integer $n \geq 2$ (*note the extra condition here, since we started with $n = 2$*), so that 5^n ends in 25. Another way to write this is $5^n = 100x + 25$ for some ineteger n. We want to show the result for $n + 1$. We have:
   $$5^{n+1} = 5 \cdot 5^n = 5 \cdot (100x + 25) = 500x + 125 = 500x + 100 + 25 = 100(5x + 1) + 25$$
   which ends in 25, so the induction step is finished.

4. Base Step: (*for $n = 4$*) $4! = 24$, while $2^4 = 16$; thus $n! > 2^n$ for $n = 4$, and the base step is finished.
   Induction Step: Assume the result is true for some positive integer $n \geq 4$, so that $n! > 2^n$. For $n + 1$ we have:
   $$(n + 1)! = (n + 1) \cdot n! \geq (n + 1) \cdot 2^n \geq 2 \cdot 2^n = 2^{n+1}$$
   where the first inequaltiy holds since $n! \geq 2^n$, and the second holds since $n + 1 \geq 2$. (This is true since $n \geq 4$, and hence $n+1 \geq 5 \geq 2$). We have shown that $(n+1)! \geq 2^{n+1}$, so the result holds for $n + 1$, and the induction step is done.

5. This is similar to problem 3.
   Base Step: $8^1 - 1 = 7$, which is clearly divisible by 7, so the base step is complete.
   Induction Step: Assume the result is true for some positive integer n, so that $8^n - 1$ is

divisible by 7. Another way to write this is $8^n = 7k + 1$ for some ineteger k. We want to show the result for $k + 1$. We have:

$$8^{n+1} - 1 = 8 \cdot 8^n - 1 = 8 \cdot (7k + 1) - 1 = 56k + 8- = 56k + 7 = 7(8k + 1)$$

which is divisible by 7, so the induction step is done.

6. Base Step: Clearly $(1 + x)^1 \geq 1 + x$, so the base step is done.
   Induction Step: Assume the result is true for some positive integer n, so that $(1+x)^n \geq 1 + nx$. We need to show the result for $n + 1$. We have:

$$\begin{aligned}(1 + x)^{n+1} &= (1 + x)(1 + x)^n \\ &\geq (1 + x)(1 + nx) \quad \text{since the result holds for } n \text{ and } 1 + x \geq 0, \text{ since } x \geq -1 \\ &= 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x \quad \text{since } x^2 \geq 0\end{aligned}$$

   therefore $(1 + x)^{n+1} \geq 1 + (n + 1)x$, so the induction step is complete.

Some Harder Problems

7. Base Step: The result holds for $n = 1$, because:

$$\frac{1}{2 \cdot 1} = \frac{1}{2} < \frac{1}{\sqrt{3}} = \frac{1}{2 \cdot 1 + 1}$$

   Induction Step: Assume the result is true for some positive integer n, so that:

$$\frac{1 \cdot 3 \cdot 5 \ldots \cdot (2n - 1)}{2 \cdot 4 \cdot 6 \ldots \cdot 2n} \leq \frac{1}{\sqrt{2n + 1}}$$

   The result for $n + 1$ is similar, except we need to multiply the left side of the above equation by $\frac{2n + 1}{2n + 2}$, while the right side by $\frac{\sqrt{2n + 1}}{\sqrt{2(n + 1) + 1}}$. So as long as we show that we multiply the left side by a smaller number, than the right side, the result will hold. Thus we just need to prove:

$$\frac{2n + 1}{2n + 2} \leq \frac{\sqrt{2n + 1}}{\sqrt{2(n + 1) + 1}}$$

   Squaring both sides we get:

$$\begin{aligned}\frac{(2n + 1)^2}{(2n + 2)^2} \leq \frac{2n + 1}{2n + 3} &\Leftrightarrow (2n + 1)(2n + 3) \leq (2n + 2)^2 \\ &\Leftrightarrow 4n^2 + 8n + 3 \leq 4n^2 + 8n + 4\end{aligned}$$

which is clearly true. Therefore the result holds for $n+1$:

$$\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}\times\frac{2n+1}{2n+2}\leq\frac{1}{\sqrt{2n+1}}\times\frac{\sqrt{2n+1}}{\sqrt{2(n+1)+1}}=\frac{1}{\sqrt{2(n+1)+1}}$$

and we are done.

8. We look at the first few terms of the sequence and try to spot a pattern. Have $a_1 = 1, a_2 = 2, a_3 = 4, a_4 = 8, \ldots$. This suggests that $a_n = 2^{n-1}$. Let us prove this using *strong induction*.
Base Step: For $n = 1$ and $n = 2$ the result holds: $a_1 = 1 = 2^{1-1}$ and $a_2 = 1 = 2^{2-1}$.
Note: In this base step, we considered two cases, and not one as usual. We need to do it for this problem since the $n+1$-st term depends on two previous terms, and not just one.
Induction Step: Let $n \geq 2$ be a positive integer, and assume the result holds for all positive integers $k \leq n$. To prove the result holds for $n+1$, we use the definition of the sequence:
$$a_{n+1} = a_n + 2a_n = 2^{n-1} + 2\cdot 2^{n-2} = 2^n = 2^{(n+1)-1}$$
where the second equality holds since the result holds for n and $n-1$, so that $a_n = 2^{n-1}, a_{n-1} = 2^{n-2}$, by the induction assumption. We have thus shown the result holds for $n+1$, and we are done.

9. We will call a positive integer n *good* if Bob can pay for an item worth n dollars with two- and five- dollar bills without change. It's clear that if n is good, then $n+2$ must be good, since Bob just pays with another two-dollar bill. So for a particular positive integer n to prove that is good, it would suffice to know that $n-2$ is good – and not just that $n-1$ is good, so we will again use strong induction.
Base Step: $n = 4$ is good: Bob can pay using two two-dollar bills.
$n = 5$ is good: Bob can pay using one five-dollar bill.
Induction Step: Let n be a positive integer, $n \geq 5$ and suppose for all integers k satisfying $4 \leq k \leq n$, k is good. We need to show that $n+1$ is good.
Since $n \geq 5$, then $n-1 \geq 4$ so $k = n-1$ satisfies $4 \leq k \leq n$ and hence $n-1$ is good. But then $n+1$ is good (remember Bob just uses an extra two-dollar bill), and therefore $n+1$ is good as well, so the induction step is finished.
Note: Make sure you understand the logic in the above argument well, and if not, re-read the solution. The induction step itself is quite simple, but the logic for setting it up is more tricky than in the previous problems.

10. Base Step: For $n = 1$ we have $3^{(2^1)} - 1 = 8$ which is divisible by $2^3 = 2^{1+2}$ but not by $2^4 = 2^{1+3}$. So the base step is done.
Induction Step: Assume the result is true for some positive integer n. Then we have $3^{(2^n)} - 1$ is divisible by 2^{n+2} but not 2^{n+3}. This is the same as saying that $3^{(2^n)} - 1 =$

$2^{n+2}k$, where k is an odd number.
For $n+1$ we have:

$$\begin{aligned}3^{(2^{n+1})}-1&=[3^{(2^n)}]^2-1\\&=(3^{(2^n)}-1)(3^{(2^n)}+1)\\&=(3^{(2^n)}-1)(3^{(2^n)}-1+2)\\&=(2^{n+2}k)(2^{n+2}k+2)\\&=2^{n+2}k\times 2\times(2^{n+1}k+1)\\&=2^{n+3}(k\times(2^{n+1}k+1))\end{aligned}$$

Since $n\geq 1$, it follows that $(2^{n+1}k+1)$ and hence $k\times(2^{n+1}k+1)$ is odd. Therefore from the above derivation we obtain that $3^{(2^{n+1})}-1$ is divisible by $2^{n+3}=2^{(n+1)+2}$ but not $2^{n+4}=2^{(n+1)+2}$, and the induction step is complete.

11. With 1 line there are 2 regions. If we draw another line, we add 2 more regions. If we draw a third line, we add 3 more regions. In general, if we have n lines already drawn, and we draw another line, then new line will intersect each of the other lines exactly once, and thus will be split up into $n+1$ disjoint segments or rays. For each of these segments or rays, we can mark off the region on the paper when only n regions were drawn, which contains this segment or ray after it is drawn. This region will be split up in exactly two regions, thus increasing the count of the regions by 1. Therefore each disjoint segment or ray creates a distinct new region, and in total $n+1$ new regions are created.
Using the above argument, we can prove by induction that the number of regions created by n lines is:

$$1+1+2+\ldots+n=1+\frac{n(n+1)}{2}$$

Base Step: For $n=1$ we have $2=1+1$ regions, so the result holds.
Induction Step: Assume the result is true for some positive integer n, so that the number of regions when n lines are drawn is:

$$1+1+2+\ldots+n$$

When we draw another line, we get $n+1$ lines, while the number of regions increases by $n+1$ (by the argument stated above), hence the number of regions is:

$$(1+1+2+\ldots+n)+(n+1)=1+1+2+\ldots+n+1$$

so the result is true for $n+1$, and the induction step is finished.

12. We will use strong induction.
Base Step: For $n=1$, $x+\dfrac{1}{x}$ is an integer by the problem condition.

For $n = 2$, we have:

$$x^2 + \frac{1}{x^2} = (x + \frac{1}{x})^2 - 2 \cdot x \cdot \frac{1}{x} = (x + \frac{1}{x})^2 - 2$$

which is an integer since $x + \frac{1}{x}$ and 2 are both integers. This completes the base step. Induction Step: Let $n \geq 2$ be an integer, and assume the result holds for all positive integers k, $k \leq n$. In particular, this means that $x^n + \dfrac{1}{x^n}$ is equal to some integer a, and $x^{n-1} + \dfrac{1}{x^{n-1}}$ is equal to some integer b. For $n + 1$ we have:

$$\begin{aligned} x^{n+1} + \frac{1}{x^{n+1}} &= (x^n + \frac{1}{x^n})(x + \frac{1}{x}) - (x^{n-1} + \frac{1}{x^{n-1}}) \\ &= a(x + \frac{1}{x}) - b \end{aligned}$$

which must be an integer, since a, b are integers, and by the problem condition, $x + \dfrac{1}{x}$ is also an integer. Thus the result is true for $n + 1$ as well, and we are done.

Hard Problems

13. Notice that for (a) and (b) m and n have greatest common divisor equal to 1, therefore on each turn a player can always make a move of replacing the number k with its divisor l strictly less than k, as long as $l > 1$, or as long as $l = 1$ and 1 has not yet appeared on the board.

    Instead of dealing with the actual numbers we will deal with the number of prime factors they have. Then, the game becomes equivalent to the following. Two numbers m and n are written on the board. On each turn a player can select a number k greater than 0 and replace it with any positive integer less than k, or replace it with 0, as long as 0 is not already written on the board. A player who cannot make a move loses.

    It immediately follows that $m = 0, n = 1$ is a losing position. Therefore, $m = 0, n \geq 2$ is a winning position (since a player replaces n with 1 and wins). Furtheremore, $m = 1, n \geq 1$ is a winning position (since a player replaces n with 0 and wins). Hence $m = 2, n = 2$ is a losing position; $m = 2, n \geq 3$ is a winning position; $m = 3, n = 3$ is a losing position, $m = 3, n \geq 4$ is a winning position. *By induction* it follows that for $k \geq 2$, $m = k, n = k$ is a losing position, while $m = k, n \geq k + 1$ is a winning position.

    (a) We are in the case of $m = 40, n = 51 \geq 41$ in the "transformed" game, thus this is a winning position and Alphonse wins.

    (b) This case is different, since now m and n have more than one divisor in common. We will deal with the original game and not make any transformations. Note that m

and n are both powers of 2, so throughout the whole game only powers of 2 can appear on the board.

We first note that the player who first writes down a number less than or equal to 2 loses. This is because if they write down 1, then 2 has not yet been written; the opponent on the next turn replaces the other number with 2 wins. (Note that this move is legal since at the start $m > 2, n > 2$ so at the time that 1 is written, the other number on the board must be greater than 2). If they write down 2, then 1 has not yet been written; the opponent on the next turn replaces the other number with 1 and wins.

Similarly, the player who first writes down a number less than or equal to 8 loses. This is because if they write down 4, the other player writes 8 – thus forcing the original player to write down a number less than or equal to 2 (note they cannot replace 8 with 4 since 4 has already appeared on the board). Similarly, if they write down 8, the other player writes down 4 and wins.

By induction it follows that if $m, n > 2^{2k-1}$ then the player who first writes down a number less than or equal to 2^{2k-1} loses for every positive integer k. Thus for the case $m = 2^{40}, n = 2^{53}$, the player to first write down a number less than or equal to 2^{39} loses. Therefore on his first turn, Alphonse replaces 2^{53} with 2^{41} and wins – because on her turn, Beryl is faced with 2^{40} and 2^{41} on the board and has to write down a number less than or equal to 2^{39}.

Note: you should always consider induction as a possible method for proving that a particular strategy in a game is a winning one – because often the objective in games is to reduce the current state of the game to a "smaller" state, which is exactly how induction works.

14. We will prove that n^p gives the remainder of n when divided by p. Since p is prime, if we manage to prove this, then we get the result, since then $n^{p-1} = \frac{n^p}{n}$ gives the remainder of $\frac{n}{n} = 1$ when divided by p. As usual, we use induction. This time, we need to only prove the result for all integers n between 1 and $p-1$, inclusive, since for any positive integer n, we have n^p gives the same remainder when divided by p, as k^p, where k is the remainder that n itself gives, when divided by p. (This is a useful fact in number theory).

    <u>Base Step</u>: $n = 1$. Then $1^p = 1$, which gives remainder 1 when divided by p. Thus the base step is done.

    <u>Induction Step</u>: Let n be a positive integer, with $1 \leq n \leq p-2$, such that the result holds for n, i.e. n^p gives remainder n when divided by p. Another way to write this is

$n^p = kp + n$ for some integer k. By the binomial theorem, for $n+1$ we have:

$$(n+1)^p = n^p + \binom{p}{1}n^{p-1} + \binom{p}{2}n^{p-2} + \dots \binom{p}{p-1}n + 1$$

For $i = 1, \dots, p-1$, the binomial coefficient:

$$\binom{p}{i} = \frac{p(p-1)\dots(p-i+1)}{i!}$$

must be divisible by p, since the numerator is divisible by p, while the denominator is not (it's a product of integers all of whom are less than p and hence cannot be divisible by p, since it is prime). This means that in the binomial expansion of $(n+1)^p$ above, all the terms involving binomial coefficients are divisible by p, and hence the remainder that $(n+1)^p$ gives when divided by p is the same as $n^p + 1$ does:

$$\begin{aligned}(n+1)^p &= n^p + \underbrace{\binom{p}{1}n^{p-1} + \binom{p}{2}n^{p-2} + \dots \binom{p}{p-1}n}_{\text{divisible by } p} + 1 \\ &= n^p + lp + 1\end{aligned}$$

for some integer l. Replacing n^p with $kp + n$ we get:

$$(n+1)^p = kp + n + lp + 1 = p(k+l) + (n+1)$$

which means that $(n+1)^p$ gives the remainder of $(n+1)$ when divided by p, and the induction step is complete.

15. Base Step: $n = 4$. We have:

$$\begin{aligned}&\frac{x_1}{x_4 + x_2} + \frac{x_2}{x_1 + x_3} + \frac{x_3}{x_2 + x_4} + \frac{x_4}{x_3 + x_1} \\ &= \frac{x_1 + x_3}{x_2 + x_4} + \frac{x_2 + x_4}{x_1 + x_3} \geq 2\sqrt{\frac{x_1 + x_3}{x_2 + x_4} \cdot \frac{x_2 + x_4}{x_1 + x_3}} = 2\end{aligned}$$

using the AM-GM inequality. This completes the base step.

Induction Step: Assume the result holds for some positive integer n; we wish to prove it for $n+1$. Let $x_1, x_2, \dots, x_{n+1}$ be arbitrary positive real numbers; we wish to show:

$$A = \frac{x_1}{x_{n+1} + x_2} + \frac{x_2}{x_1 + x_3} + \dots + \frac{x_{n+1}}{x_n + x_1} \geq 2$$

In the above expression, we can without loss of generality assume that x_{n+1} is the smallest among $x_1, x_2, \dots, x_{n+1}$. This will be used later.

By the induction assumption, we know that:

$$B = \frac{x_1}{x_n + x_2} + \frac{x_2}{x_1 + x_3} + \ldots + \frac{x_n}{x_{n-1} + x_1} \geq 2$$

If we can show that $A \geq B$, then it would immediately follow that $A \geq 2$ and we would be done. Let us try to show $A \geq B$ by looking at the difference between these two numbers:

$$\begin{aligned} A - B &= \frac{x_1}{x_{n+1} + x_2} + \frac{x_n}{x_{n-1} + x_{n+1}} + \frac{x_{n+1}}{x_n + x_1} - (\frac{x_1}{x_n + x_2} + \frac{x_n}{x_{n-1} + x_1}) \\ &= (\frac{x_1}{x_{n+1} + x_2} - \frac{x_1}{x_n + x_2}) + (\frac{x_n}{x_{n-1} + x_{n+1}} - \frac{x_n}{x_{n-1} + x_1}) + \frac{x_{n+1}}{x_n + x_1} \end{aligned}$$

The first bracket in the above expression is non-negative since $x_{n+1} + x_2 \leq x_n + x_2$, which is true because $x_{n+1} \leq x_n$ (recall that we assumed earlier than $x_{n+1} \leq x_i$ for all $i = 1, 2, \ldots n$). The second bracket is non-negative since $x_{n-1} + x_{n+1} \leq x_{n-1} + x_1$, which is true because $x_{n+1} \leq x_1$. And the last term is positive since x_{n+1}, x_n, x_1 are all positive. Thus the whole expression is non-negative, and $A - B \geq 0$.

Therefore $A \geq B$. Since $B \geq 2$, it follows that $A \geq 2$, which means that the result is true for $n + 1$ and the induction step is done.

Note: The assumption that x_{n+1} was the smallest among the terms is key here. How does one come up with such an assumption? Well, we could write out the difference $A - B$ regardless of any assumptions. Then we see, that this would be non-negative if we can get $x_{n+1} \leq x_n$ and $x_{n+1} \leq x_{n-1}$. So as long as in the sequence of numbers $x_1, x_2, \ldots, x_{n+1}, x_1, x_2$ we can find one that is less than or equal to the previous two, we are in business. This is of course always possible.