More Algebra and Some Contest Problems

A **quadratic polynomial** is a polynomial of the form $ax^2 + bx + c$ with $a \neq 0$.

The roots (solutions) of a quadratic polynomial are $\boxed{x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

1. Vieta's Theorem: $ax^2 + bx + c = a(x - x_1)(x - x_2)$ where x_1, x_2 are the roots. Therefore:
$$x_1 + x_2 = -\frac{b}{a}, x_1x_2 = \frac{c}{a}$$
2. Discriminant: The discriminant of an equation $ax^2 + bx + c$ is $D = b^2 - 4ac$. The equation has two real roots if $D > 0$, one real root if $D = 0$, and no real roots if $D < 0$.
3. Factoring: Sometimes factoring an equation simplifies it greatly.

Problems

1. Solve $(x^2 - 3x + 2)(x^2 - 3x - 4) - (x^2 - 6x + 8) = 0$.
2. Let the roots of $x^2 - 17x + 13 = 0$ be r and s. What is the value of:
(a) r^2s^2 (b) $r^2 + s^2$ (c) $r^2s + s^2r$ (d) $r^3 + s^3$
(Hint: you are not supposed to use the quadratic formula)
3. Do there exist positive integers a, b, c such that each of the equations $ax^2 + bx + c = 0, ax^2 - bx + c = 0, ax^2 + bx - c = 0, ax^2 - bx - c = 0$, has two integer roots?
4. The absolute value of the difference of the roots of the quadratic equation $x^2 + bx + c$ is 2.
(a) What is the absolute value of the difference of the roots of the equation $x^2 + 6bx + 36c$?
(b) Prove the absolute value of the difference of the roots of the equation $x^2 + 20bx + 64c$ is at least 32.
(Hint: think about how you can use Vieta's formula)
5. (Euclid 2010) Determine all real values of x such that $(x + 8)^4 = (2x + 16)^2$.
6. (Euclid 2008) Determine all $k \neq 0$ such that the equation $y = kx^2 + (5k + 3)x + (6k + 5)$ has exactly one solution.
7. (COMC 2008) Determine all real x such that
(a) $(x + 3)(x - 6) = -14$
(b) $2^{2x} - 3(2^x) - 4 = 0$
(c) $(x^2 - 3x)^2 = 4 - 3(3x - x^2)$
(Hint: for (b), (c), use substitution).
8. (Fermat 2008) For how many integers k do the graphs of $y = -\frac{1}{8}x^2 + 4$ and $y = x^2 - k$ intersect on or above the $x-$axis?

9. (COMC 2004) (a) Determine the two values of x such that $x^2 - 4x - 12 = 0$.
   (b) Determine the one value of x such that $x - \sqrt{4x + 12} = 0$.
   (c) (Hard) Determine all real c such that $x^2 - 4x - c - \sqrt{8x^2 - 32x - 8c} = 0$ has exactly two real solutions for x.

Some Nice Problems from Competitions this Year

1. (Cayley) Suppose x and y are positive real numbers such that:

$$xy = \frac{1}{9}$$
$$x(y+1) = \frac{7}{9}$$
$$y(x+1) = \frac{5}{18}$$

   What is $(x+1)(y+1)$?

2. (Cayley) Dolly, Molly, and Polly each can walk at 6 km/h. Their one motorcycle, which travels at 90 km/h, can accommodate at most two of them at once (and cannot drive by itself), Let t hours be the time taken for all three of them to reach a point 135 km away. Ignoring the time required to start, what is true about the smallest possible value of t?

   (A) $t < 3.9$ (B) $3.9 \leq t < 4.1$ (C) $4.1 \leq t < 4.3$ (D) $4.3 \leq t < 4.5$ (E) $4.5 \leq t$

3. (Fermat) Number of pairs of positive integers (p, q) with $p + q \leq 100$ such that $\frac{p + \frac{1}{q}}{q + \frac{1}{p}} = 17$ is:

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

4. (Fermat) Four numbers w, x, y, z satisfy $w < x < y < z$. Each of the six possible pairs of distinct numbers has a different sun. The four smallest sums are 1, 2, 3, and 4. What is the sum of all possible values of z?

   (A) 4 (B) $\frac{13}{2}$ (C) $\frac{17}{2}$ (D) $\frac{15}{2}$ (E) 7

5. (AMC 12) A bug travels in the coordinate plane, moving only along the lines that are parallel to the x-axis or the y-axis. Let $A = (-3, 2)$ and $B = (3, -2)$. Consider all possible paths of the bug from A to B of length at most 20. How many points with integer coordinates lie on at least one of these paths?

   (A) 161 (B) 185 (C) 195 (D) 227 (E) 255

6. (AMC 12) For two positive integers x, y, it is known that their arithmetic mean $\frac{x+y}{2}$ is a two-digit integer. Their geometric mean $\sqrt{xy}$ is obtained by reversing the digits of the arithmetic mean. What is $|x - y|$?

   (A) 24 (B) 48 (C) 54 (D) 66 (E) 70