Counting

1. **Product Rule:** If there are p ways to perform action A and q ways to perform action B then there are $\boxed{p \times q}$ ways to perform action A and action B.
   The following rule is useful when dealing with problems involving several cases:
2. **Sum Rule:** If there are p ways to perform action A and q ways to perform action B then there are $\boxed{p + q}$ ways to perform action A or action B.

   For a positive integer n, we let $\boxed{n! = 1 \times 2 \times ... \times n}$ (read "n factorial").
   By convention, $0! = 1$.
3. If there are n different objects, the number of ways to arrange them in a row is $\boxed{n!}$
4. If there are n different objects, the number of ways to select k of them (so that the order of selection matters) is $n \times (n-1) \times ... \times (n-k+1) = \boxed{\dfrac{n!}{(n-k)!}}$.
5. If there are n different objects, the number of ways to select k of them (so that order of selection does not matter) is the number obtained in the previous part, but divided by $k!$.
   Therefore the number is $\boxed{\dfrac{n!}{k!(n-k)!}}$. It is often denoted as $\dbinom{n}{k}$ and is read as "n choose k".

For many counting problems it comes down to reducing the problem to one (or more) of the above "fundamental problems". Recognizing how to do this comes with practice.

Problems

1. There were 20 people at a meeting. Every two of them shook hands with each other exactly once. How many handshakes were made?
2. All the roads in a certain country are one-way (so you can only travel in one direction). There are 4 different roads leading from a village to city A and 5 different roads leading from the same village to city B. There are 10 roads leading from city A to city B and 7 roads leading from city B to city A. There is only 1 road leading from city A to the village and 2 roads leading from city B to the village. In how many ways can a merchant leave the village, visit both cities, and come back to the village, if he must travel along exactly 3 roads in total?
3. A license plate contains 4 letters followed by 3 digits (for example $APTK$ 823). How many different license plates are there?
4. (Fermat 2010) A gumball machine that randomly dispenses one gumball at a time contains 13 red, 5 blue, 1 white, and 9 green gumballs. What is the least number of gumballs that Wally must buy to guarantee that he receives 3 gumballs of the same colour?
5. In a high school, there are 100 students in each of the four grades (grades 9, 10, 11, 12). A student government must be formed. The government must have a president, a vice-president, and four commissioners (each student can hold at most one position). In how many ways can the government be formed if (the three parts below are separate problems):

a. Any student can hold a government position.
b. One commissioner must be selected from every grade.
c. The president must be from grade 12 and the vice president must be from grade 11 or grade 12.

6. **a.** How many ways are there to arrange 30 different people in a line in front of a ticket booth?
   b. How many ways are there to seat 30 different people at a round table?
   c. Among the 30 people two are friends. How many ways are there to arrange the 30 people in a line if the two friends must stand beside each other?
   d. What if there are 3 friends that want to stand beside each other?
   e. How many different necklaces can be made from 30 different-colored marbles? (Note this is question is different from b.)
7. How many different rearrangements of letters can be made from each of the following words: PENCIL, PAPER, ERASER, NOTEBOOK, MISSISSIPPI? (Hint: First assume all letters are different and then see how many times you "overcounted" each rearrangement.)
8. (Cayley 2009) How many integers n are there with the property that the product of the digits of n is 0, where $5000 \leq n \leq 6000$?
9. How many 4-digit numbers are there, which:
   a. Contain all different digits.
   b. Contain exactly one digit 7.
   c. Contain at least one digit 7.
   d. Contain exactly one 4 and exactly one 5.
10. **a.** A child has 10 yellow marbles. He wants to take some of them to school to show to his friends. He decides to take at least one marble. How many different combinations of marbles can he take to school?
    b. Another child has 5 red marbles, 2 white marbles, and 3 green marbles. He also wants to take some of his marbles to school. He decides to take at least 1 marble of each color. How many different combinations of marbles can he take to school?
    c. A third child has only 6 marbles, but all of them are of different colors. She also wants to take some of her marbles to school, and she decides to take at least one marble in total (so she can take 1, 2, 3, 4, 5, or all 6 marbles). How many different combinations of marbles can she take to school?

Some Harder Problems

1. In how many ways can we paint the faces of a cube in 3 colors, if every color must be used at least once? (Rotations of a cube do not change the coloring).
2. For how many positive integers $x_1, x_2, ..., x_{10}$ do we have $x_1 + x_2 + ... + x_{10} = 50$? (Hint: consider 50 ones arranged in a row and convert this to a counting problem.)
3. (Pascal 2005) A number is called special if each of its digits is less than the digit to the left. For example, 5420 is special. How many special numbers are there between 200 and 700?
4. (Fermat 2007) What is the number of 3-digit positive integers a such that both a and $2a$ have only even digits? (Hint: think about column multiplication by 2. What happens when a 1 is carried?)