Counting Continued

For a positive integer n, we let $\boxed{n! = 1 \times 2 \times ... \times n}$ (read "n factorial").
By convention, $0! = 1$.

1. If there are n different objects, the number of ways to arrange them in a row is $\boxed{n!}$
2. If there are n different objects, the number of ways to select k of them (so that the order of selection matters) is $n \times (n-1) \times ... \times (n-k+1) = \boxed{\dfrac{n!}{(n-k)!}}$.
3. If there are n different objects, the number of ways to select k of them (so that order of selection does not matter) is the number obtained in the previous part, but divided by $k!$. Therefore the number is $\boxed{\dfrac{n!}{k!(n-k)!}}$. It is often denoted as $\dbinom{n}{k}$ and is read as "n choose k".

For many counting problems it comes down to reducing the problem to one (or more) of the above "fundamental problems". Recognizing how to do this comes with practice.

Problems

1. How many different rearrangements of letters can be made from each of the following words: PENCIL, PAPER, ERASER, NOTEBOOK, MISSISSIPPI? (Hint: First assume all letters are different and then see how many times you "overcounted" each rearrangement.)
2. (Cayley 2009) How many integers n are there with the property that the product of the digits of n is 0, where $5000 \leq n \leq 6000$?
3. How many 4-digit numbers are there, which:
   a. Contain all different digits.
   b. Contain exactly one digit 7.
   c. Contain at least one digit 7.
   d. Contain exactly one 4 and exactly one 5.
4. **a.** A child has 10 yellow marbles. He wants to take some of them to school to show to his friends. He decides to take at least one marble. How many different combinations of marbles can he take to school?
   b. Another child has 5 red marbles, 2 white marbles, and 3 green marbles. He also wants to take some of his marbles to school. He decides to take at least 1 marble of each color. How many different combinations of marbles can he take to school?
   c. A third child has only 6 marbles, but all of them are of different colors. She also wants to take some of her marbles to school, and she decides to take at least one marble in total (so she can take 1, 2, 3, 4, 5, or all 6 marbles). How many different combinations of marbles can she take to school?
5. An ant is at the point $(0,0)$ in the coordinate plane. It needs to get to the point $(3,4)$. It can only travel along the grid lines, and it can only move up or to the right. In how many ways can it get to the point $(3,4)$? (Hint: represent every path in terms of individual steps).

6. A waiter must seat 6 women and 6 men at a round table. No two women are allowed to sit beside each other, and no two men are allowed to sit beside each other. In how many ways can the waiter seat the 12 people?
7. (Pascal Math Contest 2005, #19) A number is called special if each of its digits is less than the digit to the left. For example, 5420 is special. How many special numbers are there between 200 and 700?
8. (Cayley Math Contest 2010, #19) What is the number of 3-digit positive integers that have exactly one even digit?
9. (a) For how many positive integers $x_1, x_2, ..., x_{10}$ do we have $x_1 + x_2 + ... + x_{10} = 50$? (Hint: consider 50 ones arranged in a row and convert this to a counting problem.)
   (b) For how many non-negative integers $x_1, x_2, ..., x_{10}$ do we have $x_1 + x_2 + ... + x_{10} = 50$?
10. (Fermat Math Contest 2007, #15) What is the number of 3-digit positive integers a such that both a and $2a$ have only even digits? (Hint: think about column multiplication by 2. What happens when a 1 is carried?)

Some Harder Problems

1. (AIME 2006) A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:
   (a) Any cube may be the bottom cube in the tower.
   (b) The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.
   Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?
2. (AIME 2011) Ed has five identical green marbles and a large supply of identical red marbles. He arranges the green marbles and some of the red marbles in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves equals the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let m be the maximum number of red marbles for which Ed can make such an arrangement, and let N be the number of ways in which Ed can arrange the $m + 5$ marbles to satisfy the requirement. Find the remainder when N is divided by 1000.
3. (AIME 2011) Six men and some number of women stand in a line in random order. Let p be the probability that a group of at least four men stand together in the line, given that every man stands next to at least one other man. Find the least number of women in the line such that p does not exceed 1 percent.

Answers - SPOILER ALERT

1. $6!, \frac{5!}{2!}, \frac{6!}{2!2!}, \frac{8!}{3!}, \frac{11!}{4!4!2!}$
2. 272
3. 4536, 1848, 6312, 588
4. 10, 30, 63
5. 35
6. $\frac{11!}{5!6!}$
7. 20
8. 350
9. $\frac{49!}{9!40!}, \frac{59!}{9!50!}$
10. 18

Harder problems:

1. 458
2. 3
3. 594