Counting

1. (AMC 2010) Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

2. **a.** An ant is at the point $(0, 0)$ in the coordinate plane. It needs to get to the point $(3, 4)$. It can only travel along the grid lines, and it can only move up or to the right. In how many ways can it get to the point $(3, 4)$? (Hint: represent every path in terms of individual steps).
b. (AMC 2010) A 16-step path is to go from $(-4, -4)$ to $(4, 4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ at each step?

3. (AMC 2010) The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

4. (AIME 2006) A sequence is defined as follows $a_1 = a_2 = a_3 = 1$, and, for all positive integers n, $a_{n+3} = a_{n+2} + a_{n+1} + a_n$. Given that $a_{28} = 6090307, a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when $\sum_{k=1}^{28} a_k$ is divided by 1000.

5. (AIME 2006) A collection of 8 cubes consists of one cube with edge-length k for each integer $k, 1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:
(a) Any cube may be the bottom cube in the tower.
(b) The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.
Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

6. (AIME 2007) Let S be a set with six elements. Let P be the set of all subsets of S. Subsets A and B of S, not necessarily distinct, are chosen independently and at random from P. the probability that B is contained in at least one of A or $S - A$ is $\frac{m}{n^r}$ where m, n, r and are positive integers, n is prime, and m and n are relatively prime. Find $m + n + r$. (The set $S - A$ is the set of all elements of S which are not in A.)

7. (AIME 2010) Let N be the number of ways to write 2010 in the form $2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0$, where the a_i's are integers, and $0 \leq a_i \leq 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N.

8. (AIME 2011) Ed has five identical green marbles and a large supply of identical red marbles. He arranges the green marbles and some of the red marbles in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves equals the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let m be the maximum number of red marbles for which Ed can make such an arrangement, and let N be the number of ways in which Ed can arrange the $m + 5$ marbles to satisfy the requirement. Find the remainder when N is divided by 1000.

9. **a.** Ten people are numbered from 1 to 10 and must sit in chairs numbered from 1 to 10 (so that each person is assigned a specific chair). How many ways are there for the people to sit so that no person sits in the chair assigned to him?
   b. (AIME 2011) Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

10. (AIME 2011) Six men and some number of women stand in a line in random order. Let p be the probability that a group of at least four men stand together in the line, given that every man stands next to at least one other man. Find the least number of women in the line such that p does not exceed 1 percent.

Answers - SPOILER ALERT

1. $\frac{37}{56}$
2. (b) 1698
3. 42
4. 834
5. 458
6. 710
7. 202
8. 3
9. (b) 97
10. 594