Euclid Preparation

The Euclid contest is on Wednesday, April 11. Some advice on preparing for and writing the competition:

- Review standard contest math, with a special focus on algebra:
  - *Number Theory*: divisibility and remainders, digits of numbers and divisibility, primes
  - *Geometry*: coordinate, Euclidean, trigonometry, Sine and Cosine Laws
  - *Algebra*: sequences and series, equations and systems of equations, functions, inequalities, logarithms and exponents, parabolas
  - *Combinatorics*: counting, probability, induction, games
- CEMC has posted some useful preparation materials online: `http://cemc.uwaterloo.ca/contests/euclid_eWorkshop.html`
- Do A LOT of problems. The best way is to do past competitions, which can be found here: `http://cemc.uwaterloo.ca/contests/past_contests.html`
  If that's not enough, do some COMC problems found here:
  `http://cms.math.ca/Competitions/COMC/`
  and/or AIME problems: `http://www.artofproblemsolving.com/Forum/resources.php?c=182&cid=45&sid=21e059b15d32fe1842bc7ca216046a49`. Note the style and difficulty of the problems on these other two competitions is similar, but not the same as the Euclid.
- When doing past contests, make sure to time yourself. Also simulate the competition environment - do it in a quiet place for the full 2.5 hours in one sitting, and write up the solutions as if you are officially writing the competition.
- After doing a past contest, go through all the solutions (including for the problems you solved). Sometimes it helps to read the official solution even if you solved a problem correctly, since then you may get an idea on how to write solutions more efficiently. It helps to learn how to write up complete solutions in a concise and rigorous manner - this will make sure to get you full marks and also save precious time.
- After doing a few past contests you will be able to estimate a reasonable target score on the competition. Develop a plan on how many problems you plan to solve and how much time you plan to spend on them.
- Get a good night sleep before the day of the contest.
- Make sure you bring all the necessary supplies for the contest - pens, pencils, eraser, ruler, compass, protractor, calculator. Water and chocolate are also helpful. Most importantly - **bring a watch**. Make sure to keep track of time throughout the whole contest.
- Leave some time before the end of the competition to check all your solutions and answers, especially in problems involving a lot of calculations.
- Even if you can't solve a problem, write down all the ideas and partial progress that you have on it. Euclid markers tend to be generous with part marks.

Some Past Euclid Problems - Warm-Up

1. (Euclid 2007.1a) If the point $(a-1, a+1)$ lies on the line $y = 2x - 3$, what is the value of a?
2. (Euclid 2006.1b) If the lines $px = 12$ and $2x + qy = 10$ intersect at $(1, 1)$, what is the value of $p + q$?
3. (Euclid 2007.2b) Suppose $0° < x < 90°$ and $2\sin^2 x + \cos^2 x = \frac{25}{16}$. What is the value of $\sin x$?
4. (Euclid 2006.2b) When a decimal point is placed between the digits of the two-digit integer n, the resulting number is equal to the average of the digits of n. What is the value of n?
5. (Euclid 2006.3a) Determine the coordinates of the vertex of the parabola $y = (x-20)(x-22)$.
6. (Euclid 2007.3a) The first term in a sequence is 2007. Each term, starting with the second, is the sum of the cubes of the digits of the previous term. What is the 2007th term?
7. (Euclid 2007.4a) Determine all values of x for which $2 + \sqrt{x-2} = x - 2$.
8. (Euclid 2006.5a) If a is chosen randomly from the set $\{1, 2, 3, 4, 5\}$ and b is chosen randomly from the set $\{6, 7, 8\}$, what is the probability that a^b is an even number?

Easy Problems

1. (Euclid 2007.4b) A parabola intersects the x-axis at $A(-3, 0)$ and $B(3, 0)$ and has its vertex at C lying on the y-axis below the x-axis. The area of $\triangle ABC$ is 54. Determine the equation of the parabola. Explain how you got your answer.
2. (Euclid 2007.6a) The Little Prince lives on a spherical planet which has a radius of 24 km and centre O. He hovers in a helicopter (H) at a height of 2 km above the surface of the planet. From his position in the helicopter, what is the distance, in kilometres, to the furthest point on the surface of the planet that he can see?
3. (Euclid 2006.2c) The average of three positive integers is 28. When two additional integers, s and t, are included, the average of all five integers is 34. What is the average of s and t?
4. (Euclid 2007.7a) Determine all values of x for which $(\sqrt{x})^{\log_{10} x} = 100$.
5. (Euclid 2007.8a) In a 4×4 grid, three coins are randomly placed in different squares. Determine the probability that no two coins lie in the same row or column.
6. (Euclid 2006.6a) Suppose that, for some angles x and y, $\sin^2 x + \cos^2 y = \frac{3}{2}a$ and $\cos^2 x + \sin^2 y = \frac{1}{2}a^2$. Determine the possible value(s) of a.
7. (Euclid 2006.7a) The sequence $2, 5, 10, 50, 500, \ldots$ is formed so that each term after the second is the product of the two previous terms. The 15th term ends with exactly k zeroes. What is the value of k?
8. (Euclid 2006.8a) If $\log_2 x - 2\log_2 y = 2$, determine y as a function of x, and sketch a graph of this function.

Medium Problems

1. (Euclid 2006.5b) A bag contains some blue and some green hats. On each turn, Julia removes one hat without looking, with each hat in the bag being equally likely to be chosen. If it is

green, she adds a blue hat into the bag from her supply of extra hats, and if it is blue, she adds a green hat to the bag. The bag initially contains 4 blue hats and 2 green hats. What is the probability that the bag again contains 4 blue hats and 2 green hats after two turns?

2. (Euclid 2007.5b) $A(0, a)$ lies on the y-axis above $D(0, 1)$. B has coordinates $(1, 0)$, C has coordinates $(3, 2)$, and O is the origin. If the triangles AOB and BCD have the same area, determine the value of a. Explain how you got your answer.
3. (Euclid 2007.7b) The line segment FCG passes through vertex C of square $ABCD$, with F lying on AB extended through B and G lying on AD extended through D. Prove that $\frac{1}{AB} = \frac{1}{AF} + \frac{1}{AG}$.
4. (Euclid 2006.6b) $ABCD$ is a quadrilateral in which $\angle A + \angle C = 180°$, $AD = 5, AB = 6, DB = 7, BC = 4$. What is the length of CD?
5. (Euclid 2007.8b) The area of $\triangle ABC$ is 1. Trapezoid $DEFG$ is constructed so that B, G, F, C lie on a line in this order, DE is parallel to BC, EF is parallel to AB and DG is parallel to AC. Determine the maximum possible area of trapezoid $DEFG$.
6. (Euclid 2006.7b) Suppose that a, b, c are three consecutive terms in an arithmetic sequence. Prove that $a^2 - bc$, $b^2 - ac$, and $c^2 - ab$ are also three consecutive terms in an arithmetic sequence.
7. (Euclid 2005.7b) The parabola $y = -\frac{1}{4}(x - r)(x - s)$ intersects the axes at three points $(-k, 0), (0, 3k), (3k, 0)$. The vertex of this parabola is the point V. Determine the value of k and the coordinates of V.

Hard Problems

1. (Euclid 2007.9) The parabola $y = f(x) = x^2 + bx + c$ has vertex P and the parabola $y = g(x) = -x^2 + dx + e$ has vertex Q, where P and Q are distinct points. The two parabolas also intersect at P and Q.
(a) Prove that $2(e - c) = bd$. (b) Prove that the line through P and Q has slope $\frac{1}{2}(b + d)$ and $y-$intercept $\frac{1}{2}(c + e)$.
2. (Euclid 2006.9) Define $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k.
(a) Determine all real numbers k for which $f(x)$ is constant for all values of x.
(b) If $k = -0.7$, determine all solutions to the equation $f(x) = 0$.
(c) Determine all real numbers k for which there exists a real number c such that $f(c) = 0$.
3. (Euclid 2006.10) Points $A_1, A_2, ..., A_N$ are equally spaced around the circumference of a circle and $N \geq 3$. Three of these points are selected at random and a triangle is formed using these points as its vertices.
(a) If $N = 7$, what is the probability that the triangle is acute?
(b) If $N = 2k$ for some positive integer $k \geq 2$, determine the probability that the triangle is acute.
(c) If $N = 2k$ for some positive integer $k \geq 2$, determine all possible values of k for which the probability that the triangle is acute can be written in the form a $\frac{a}{2007}$ for some positive integer a.

4. (Euclid 2007.10) YXZ is an angle with vertex X. A circle is tangent to XY at Y and to XZ at Z. Point T is chosen on the minor arc YZ and a tangent to the circle is drawn at T, cutting XY at V and XZ at W. Prove that the perimeter of $\triangle VXW$ is independent of the position of T.
   (b) BAC is an angle with vertex A. $AB = 10, BC = 14, AC = 16$, and M is the midpoint of BC. Various lines can be drawn through M, cutting AB (possibly extended) at P and AC (possibly extended) at Q. Determine, with proof, the minimum possible perimeter of $\triangle APQ$.