# Games

Today we will be playing mathematical games. :)
In all of the games we will be dealing with there are 2 players. Because these are mathematical games, one of the players always has a winning strategy. Your goal is to figure out who has the winning strategy and what is.

1. **Winning/Losing Position:** A *winning position* is a position such that if a player is in this position, they can guarantee to win the game. The most important idea here is that if a player is in a certain position and regardless of which move he makes, his opponent will be in a winning position, then that certain position is a *losing position.* On the other hand, if a player is in a certain position, and can always make a move so that his opponent will be in a losing position, then that certain position is a winning position.

2. **Work Backwards:** Think about when the game ends. What must be the move(s) that happened right before the game ends? You can figure out some winning and losing positions, and from there work your way backwards to figure out if the starting position is winning or losing.

3. **Symmetry:** A very simple strategy is to copy or almost copy your opponent's moves. Here is an example. Two players take turns placing identical coins on a round table. The coins cannot overlap. A player loses if they can't make a move. Who can guarantee to win?
*Solution:* The first player should place a coin in the exact center of the table. Then she places every next coin in the position that is symmetric with respect to the center to the position of the last coin placed by her opponent.

4. Play the game several times with a friend (or with yourself). Then you can figure out some interesting facts about the game that could help you find the winning strategy. If the game seems complicated, play a simpler game.

### Some Interesting Games:

In all of these games there are two players who take turns making the moves. You have to find which of the players (the first or the second) has the winning strategy and what it is.

1. There are 10 circles drawn in a row on a sheet of paper. On every move, the player can cross out 1, 2, or 3 circles among the circles that have not yet been crossed out. The player who cannot make a move loses.

2. **a.** There are 2 piles of stones, each with 10 stones. On every move the player can take any number of stones from one of the piles and throw those stones away. The player who cannot make a move loses.
**b.** Same game as in (a), but now there are 3 piles of stones.

3. A $5 \times 7$ rectangle is cut out of a sheet of grid paper, so that it is made up of 35 unit squares. Two players take turns cutting the rectangle into smaller rectangles. On every move the player can select any of the remaining rectangles that is larger than a unit square, and cut it into two smaller rectangles by cutting along the grid lines. The player who cannot make a move loses.

4. There is a rook standing in the bottom left corner of a $8 \times 8$ chessboard. On every move, a player can move the rook any positive number of squares up or any number of positive squares to the right. The player who cannot make a move loses.

5. Greg and Heather take turns saying positive integers. Heather says a number that is at most 10. Then Greg says a number that is greater than the last number Heather said by at most 10. Then Heather says a number that is greater than the last number Greg said by at most 10, etc. (For example if Greg says 32, Heather can say 41 but not 43). The player who says 100 wins.

6. **a.** Two players take turns placing bishops on an $8 \times 8$ chessboard. A bishop can be placed only if it does not attack any of the bishops already placed. The player who cannot make a move loses.
   **b.** Two players take turns placing kings on an $7 \times 7$ chessboard. A king can be placed only if it does not attack any of the kings already placed. The player who cannot make a move loses.

7. There are 200 pennies in a pile. On every move a player can take at least one and at most half of the remaining pennies. The player who takes the last penny wins.

8. The numbers 1, 2, 3, ..., 30, are written in a row on the board. On every move a player can place a "+" or a "-" between two consecutive numbers (as long as there is no plus or minus between them yet). Eventually all the spaces between the numbers are filled with pluses and minuses. The resulting expression is calculated to get a number. If it is even, the first player wins. Otherwise, the second player wins.

9. Two players are playing two-move chess. The game is played in the same way as normal chess, except every time a player must make 2 moves instead of one. Explain why the first player can never lose. (Hint: Assume that the second player has a winning strategy. Can the first player make moves so that the second player "becomes" the first player?)

10. Twelve 3s are written on the board in a row. On every move a player can place a "+" or a "$\times$" between two consecutive numbers (as long as there is no plus or multiplication sign between them yet). Eventually all the spaces between the numbers are filled with addition and multiplication signs. The resulting expression is calculated to get a number. If it is even, the first player wins. Otherwise, the second player wins.

11. The numbers 1, 2, 3, ..., 100 are written on the board. On every move a player can erase one of the remaining numbers on the board. The game is played until only 2 numbers remain. If their sum is an integer multiple of 3, the first player wins, otherwise the second player wins.

12. There are 5 buckets arranged in a row. Every bucket contains a marble. On every move a player can select one of the buckets (except the rightmost bucket) and move all the marbles in the bucket to the neighboring bucket to the right. The player who cannot make a move loses.

13. a. A coin is placed in the bottom left corner of a $8 \times 8$ chessboard. On every move a player can move the coin one square up, down, right, or left, as long as the coin is moved to a square on the board where it has not been before. A player who cannot make a move loses.
    b. Same problem as in (a), but with a $9 \times 9$ chessboard.
    (Hint: divide the board into dominoes).

14. Two players take turns taking stones from a pile of 10 million stones. On every move a player must take $p^n$ stones, where $p$ can be any prime and $n$ can be any non-negative integer (chosen by the player making the move). The person who takes the last stone wins.