Geometry

We will focus on some "fundamental" approaches to solving geometry problems:

1. Angle Chasing.
2. Finding congruent and similar triangles.
3. Using properties of circles, in particular cyclic quadrilaterals and power point.

Recall the following geometric facts. If you already know these, feel free to start on the problems.

Angles, Triangles

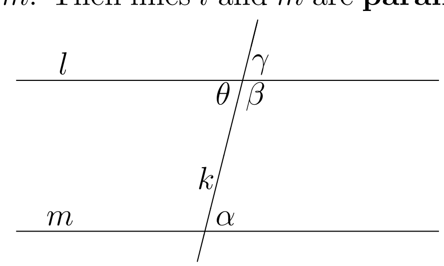
1. Vertical angles are equal. Sum of angles in a triangle is 180°.
2. Congruent triangles: SAS, ASA, SAA, SSS. Not SSA!
3. Similar triangles: SSS, AA, SAS.
4. $\triangle ABC$ is isosecles $\Leftrightarrow AB = BC \Leftrightarrow \angle A = \angle C \Leftrightarrow$ at least two of $\{$median BM, altitude BH, angle bisector $BD\}$ coincide.
5. Let ABC be an angle, and BD be its angle bisector. Let X be a point on BD. If we drop perpendiculars XK, XL to BA, BC, respectively, then $XK = XL$.
6. Let AB be a line segment and let X be any point on the perpendicular bisector of AB. Then $XA = XB$.
7. If AD is the internal angle bisector of $\angle A$ in $\triangle ABC$, then $\frac{DB}{DC} = \frac{AB}{AC}$. The same holds if AD is the external angle bisector.
8. Altitudes are concurrent, medians are concurrent, angle bisectors are concurrent, perpendicular angle bisectors are concurrent.

Parallel lines, Special Quadrilaterals

We are given lines l and m and a line k intersecting l and m. Then lines l and m are **parallel** if and only if:

a. Angles α and θ are equal, or
b. Angles α and β add to 180°, or
c. Angles α and γ are equal.

If l and m are parallel, we write $l||m$.



1. $ABCD$ is a parallelogram $\Leftrightarrow AB||CD$, $\angle A = \angle C \Leftrightarrow AB||CD$, $AB = CD \Leftrightarrow$ $AB = CD$, $BC = AD \Leftrightarrow \angle A = \angle C$, $\angle B = \angle D \Leftrightarrow$ Diagonals bisect each other.
2. $ABCD$ is a rectangle $\Leftrightarrow$ all its angles are $90^\circ \Leftrightarrow ABCD$ is a parallelogram and its diagonals are equal.
3. $ABCD$ is a rhombus $\Leftrightarrow AB = BC = CD = DA \Leftrightarrow ABCD$ is a parallelogram and diagonals are perpendicular to each other $\Leftrightarrow AC$ is angle bisector of $\angle A, \angle C$; BD is angle bisector of $\angle B, \angle D$.

Properties of Circles

1. Let O be a centre of a circle, and A, B be two different points on this circle. Let M be the midpoint of AB. Then OM is perpendicular to AB.
2. Let O be the centre of a circle, and A, B be two points on the circle. Let C be a point on the circle so that C and O lie on the same side of line AB. Then $\angle AOB = 2\angle ACB$.
3. If AB is the diameter of a circle, and C is a point on the circle then $\angle ACB = 90°$.
4. If four points A, B, C, D (in this order) lie on a circle, they are called *concyclic*, and $ABCD$ is called a *cyclic quadrilateral*. $ABCD$ is cyclic $\Leftrightarrow \angle ACB = \angle ADB \Leftrightarrow \angle ABC + \angle CDA = 180°$.

Let l be a line, and w be a circle with centre O. We say the line l is *tangent* to circle w at a point A, if l is perpendicular to OA; the line l is called the *tangent* to the circle w at A.

1. Let P be a point outside a circle. From this point we draw two tangents to the circle (it is easy to see there are only two). Let these two tangents be tangent to the circle at points A and B. Then $PA = PB$.
2. Let a circle with centre O be tangent to a line at A. Let C, D be two points on the circle, so that C, D, A are on the circle in this order. Let B be a point on the line so that C, B lie on different sides of AD. Then $\angle DCA = \angle DAB$.

Power of a Point: Suppose $ABCD$ is a cyclic quadrilateral and AB intersects CD at P. Then $PB \cdot PA = PC \cdot PD$. Suppose AC intersects BD at Q. Then $AQ \cdot QC = BQ \cdot QD$.

Exercise: Prove the above theorem. (Hint: find some similar triangles.)
If you want some training materials at the olympiad level, here are two excellent resources:

1. *Geometry Unbound* by Kiran Kedlaya:
`http://www-math.mit.edu/~kedlaya/geometryunbound/`.
2. Yufei Zhao's olympiad website: `http://web.mit.edu/yufeiz/www/olympiad.html`; in particular: `http://web.mit.edu/yufeiz/www/olympiad/geolemmas.pdf`.

1 Problems

1. Let ABC be a triangle. Extend line AB through B and mark a point D on the extended line. Then $\angle DBC$ is called an *exterior angle* of triangle ABC. Show that $\angle DBC = \angle BAC + \angle BCA$.
2. Let $ABCD$ be a cyclic quadrilateral. Let E be a point on the extension of AB through B. Prove that $\angle EBC = \angle CDA$.
3. Let AB be a line, and M, N be two points on the same side of AB. Let MK, NL be perpendiculars from M, N so line AB. If $MK = NL$, show that MN is parallel to AB.

4. Let P be a point outside a circle ω. Let PC be a tangent to ω, and let a line through P intersect ω at A, B. Prove that $PC^2 = PA \cdot PB$.
5. (COMC 2000) In $\triangle ABC$ points D, E, F are on sides BC, CA, AB, respectively. $\angle AFE = \angle BFD, \angle BDF = \angle CDE, \angle CED = \angle AEF$. Show that $\angle BDF = \angle BAC$.
6. Prove the perpendicular bisectors in a triangle are concurrent (hint: use property 6 in the Triangles section).
7. Let AD, BE be the altitudes in $\triangle ABC$. (For simplicity assume $\triangle ABC$ acute, so that D, E lie on sides BC, AC; however the properties below will still hold true without this restriction).
(a) Prove $\triangle DEC \sim \triangle ABC$.
(b) Construct a point F on AB such that $\angle AEF = \angle DEC$. Prove that $BFEC$ is cyclic, and therefore $CF \perp AB$.
(c) Let H be the intersection of AD, BE. Prove that $\angle BHD = \angle C = \angle BFD$. Conclude that C, H, F are collinear. This proves that the altitudes are concurrent.
8. Let $ABCD$ be a square with side length 1. A point E is constructed outside of this square so that triangle AEB is equilateral. What is the radius of the circle that passes through the points E, C, D? (Hint: Consider point F inside the square so that triangle CFD is equilateral.)
9. *Miquel's Theorem:* Let ABC be a triangle. Let D, E, F be points on sides BC, CA, AB, respectively. Let w_1 be the circle passing through A, F, E; w_2 be the circle passing through B, F, D; w_3 be the circle passing through C, D, E. Show that these three circles intersect at the same point. (Hint: look at the point of intersection of two of these circles and show it lies on the third circle).
10. Let A, B, C, D be points on a circle, in this order. If triangle ABC is equilateral ($AB = BC = CA$) show that $BD = AD + CD$.
(Hint: Extend CD through D and mark a point E so that $EC = BD$. Prove $BD = ED$.)
11. (Euclid 2006) Let AB, BC be chords of the circle with $AB < AC$. Let D be the point on the circle such that $AD \perp BC$ and E the point on the circle such that $DE || BC$. Show that $\angle EAC + \angle ABC = 90°$.
12. (COMC 2007) Let $CBAD$ be a trapezoid with $BA || CD$, $AB \perp BC$. Assume $BA = 9, BC = 24, CD = 18$, and let BD, CA intersect at E.
(a) Prove $DE : EB = 2 : 1$. (b) Find the area of $\triangle DEC$. (c) Find the area of $\triangle DAE$.
13. (COMC 2011) $BDEC$ is a cyclic quadrilateral, inscribed in a circle ω. BC is a diameter of ω. BD and CE intersect at a point A. $BC = \sqrt{901}$, $BD = 1$, $DA = 16$. What is the length of EC?
14. (COMC 2005) We are given a semicircle with diameter AB. A point P is chosen on AB, and points D, E on the semi-circle so that $\angle PDO = \angle EDO$; $\angle DEO = \angle OEB$, and $DP \perp AB$. Find $\angle DOP$.

15. Let ABC be a triangle, and w a circle passing through A, B, C. Let the angle bisectors of angles $\angle BAC, \angle ACB, \angle ABC$ intersect at point I. Let AI intersect the circle w at M. Prove that $MB = MI = MC$.
16. (Euclid 2010) Points A, B, P, Q, C, D lie on a line in this order. The semicircle with diameter AC has centre P, and the semicircle with diameter BD has centre Q. The semicircles intersect at R. If $\angle PRQ = 40°$, find $\angle ARD$.
17. (University of Toronto Math Club) Points P, Q are on sides AB, BC of a square $ABCD$ so that $BP = BQ$. Let S be a point on PC so that BS is perpendicular to PC. Find $\angle QSD$.
18. *Archimedes' Broken Chord Theorem:* Let A, P, B be three points on a circle in this order, so that $AP = PB$. Let C be a point on the circle between P and B, so that C and A are on different sides of line PB. Let M be a point on AC such that PM is perpendicular to AC. Show that $AM = MC + CB$. (Hint: construct point C' on AC so that $C'M = MC$. Now prove that $AC' = CB$.)
19. In a triangle ABC, $\angle ABC = 120°$, $\angle BAC = 40°$. The line AB is extended through B to a point D so that $AD = BC + 2AB$. Find $\angle DCA$. (Hint: let M be such that $DM = AB$.)
20. (Euclid 2009) Let B be a point outside a circle ω with centre O and radius r. Let BA be a tangent from B to ω. Let C be a point on the circle, and D be a point inside the circle so that B, C, D lie on a line (in this order). Assume $OD = DC = CB$. Prove that $DB^2 + r^2 = BA^2$.
(Hint: Extend BD through D).
21. (APMO 2010) Let ABC be a triangle with $\angle BAC \neq 90°$. Let O be the circumcenter of $\triangle ABC$ and ω the circumcircle of $\triangle BOC$. ω intersects line segment AB at P different from B, and line segment AC at Q difference from C. Let ON be the diameter of ω. Prove that $APNQ$ is a parallelogram.
22. (APMO 2005) Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter (intserction of angle bisectors), and H the orthocenter (intersection of altitudes) of triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.
23. (CMO 2011) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of $\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.
24. (CMO 2000) Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB, \angle ABD = 2\angle CDB$, $AB = CB$. Prove $AD = CD$.
25. (IMO SL 1997) A triangle ABC has circumcircle ω. The angle bisectors of $\angle A, \angle B, \angle C$ intersect ω again at points K, L, M respectively. Let R be a point on side AB. A point P is such that RP is parallel to AK and BP is perpendicular to BL. A point Q is such that RQ is parallel to BL and AQ is perpendicular to AK. Prove that KP, LQ, MR have a point in common.