Geometry

We will focus on some "fundamental" approaches to solving geometry problems:

1. Angle Chasing.
2. Finding congruent and similar triangles.
3. Using properties of circles, in particular cyclic quadrilaterals and power point.

Recall the following geometric facts. If you already know these, feel free to start on the problems.

Angles, Triangles

1. Vertical angles are equal. Sum of angles in a triangle is 180°.
2. Congruent triangles: SAS, ASA, SAA, SSS. Not SSA!
3. Similar triangles: SSS, AA, SAS.
4. \( \triangle ABC \) is isosceles \( \iff \) \( AB = BC \iff \angle A = \angle C \iff \) at least two of \{median \( BM \), altitude \( BH \), angle bisector \( BD \}\} coincide.
5. Let \( ABC \) be an angle, and \( BD \) be its angle bisector. Let \( X \) be a point on \( BD \). If we drop perpendiculars \( XK, XL \) to \( BA, BC \), respectively, then \( XK = XL \).
6. Let \( AB \) be a line segment and let \( X \) be any point on the perpendicular bisector of \( AB \). Then \( XA = XB \).
7. If \( AD \) is the internal angle bisector of \( \angle A \) in \( \triangle ABC \), then \( \frac{DB}{DC} = \frac{AB}{AC} \). The same holds if \( AD \) is the external angle bisector.
8. Altitudes are concurrent, medians are concurrent, angle bisectors are concurrent, perpendicular angle bisectors are concurrent.

Parallel lines, Special Quadrilaterals

We are given lines \( l \) and \( m \) and a line \( k \) intersecting \( l \) and \( m \). Then lines \( l \) and \( m \) are parallel if and only if:

a. Angles \( \alpha \) and \( \theta \) are equal, or
b. Angles \( \alpha \) and \( \beta \) add to 180°, or
c. Angles \( \alpha \) and \( \gamma \) are equal.

If \( l \) and \( m \) are parallel, we write \( l \parallel m \).

1. \( ABCD \) is a parallelogram \( \iff AB \parallel CD, \angle A = \angle C \iff AB \parallel CD, AB = CD \iff AB = CD, BC = AD \iff \angle A = \angle C, \angle B = \angle D \iff \) Diagonals bisect each other.
2. \( ABCD \) is a rectangle \( \iff \) all its angles are 90° \( \iff ABCD \) is a parallelogram and its diagonals are equal.
3. \( ABCD \) is a rhombus \( \iff AB = BC = CD = DA \iff ABCD \) is a parallelogram and diagonals are perpendicular to each other \( \iff AC \) is angle bisector of \( \angle A, \angle C \); \( BD \) is angle bisector of \( \angle B, \angle D \).
Properties of Circles

1. Let \( O \) be a centre of a circle, and \( A, B \) be two different points on this circle. Let \( M \) be the midpoint of \( AB \). Then \( OM \) is perpendicular to \( AB \).

2. Let \( O \) be the centre of a circle, and \( A, B \) be two points on the circle. Let \( C \) be a point on the circle so that \( C \) and \( O \) lie on the same side of line \( AB \). Then \( \angle AOB = 2 \angle ACB \).

3. If \( AB \) is the diameter of a circle, and \( C \) is a point on the circle then \( \angle ACB = 90^\circ \).

4. If four points \( A, B, C, D \) (in this order) lie on a circle, they are called \( \text{concyclic} \), and \( ABCD \) is called a \( \text{cyclic quadrilateral} \). \( ABCD \) is cyclic \( \iff \angle ACB = \angle ADB \iff \angle ABC + \angle CDA = 180^\circ \).

Let \( l \) be a line, and \( w \) be a circle with centre \( O \). We say the line \( l \) is \( \text{tangent} \) to circle \( w \) at a point \( A \), if \( l \) is perpendicular to \( OA \); the line \( l \) is called the \( \text{tangent} \) to the circle \( w \) at \( A \).

1. Let \( P \) be a point outside a circle. From this point we draw two tangents to the circle (it is easy to see there are only two). Let these two tangents be tangent to the circle at points \( A \) and \( B \). Then \( PA = PB \).

2. Let a circle with centre \( O \) be tangent to a line at \( A \). Let \( C, D \) be two points on the circle, so that \( C, D, A \) are on the circle in this order. Let \( B \) be a point on the line so that \( C, B \) lie on different sides of \( AD \). Then \( \angle DCA = \angle DAB \).

**Power of a Point:** Suppose \( ABCD \) is a cyclic quadrilateral and \( AB \) intersects \( CD \) at \( P \). Then \( PB \cdot PA = PC \cdot PD \). Suppose \( AC \) intersects \( BD \) at \( Q \). Then \( AQ \cdot QC = BQ \cdot QD \).

**Exercise:** Prove the above theorem. (Hint: find some similar triangles.)

If you want some training materials at the olympiad level, here are two excellent resources:

1. *Geometry Unbound* by Kiran Kedlaya:


1 **Problems**

1. Let \( ABC \) be a triangle. Extend line \( AB \) through \( B \) and mark a point \( D \) on the extended line. Then \( \angle DBC \) is called an \( \text{exterior angle} \) of triangle \( ABC \). Show that \( \angle DBC = \angle BAC + \angle BCA \).

2. Let \( ABCD \) be a cyclic quadrilateral. Let \( E \) be a point on the extension of \( AB \) through \( B \). Prove that \( \angle EBC = \angle CDA \).

3. Let \( AB \) be a line, and \( M, N \) be two points on the same side of \( AB \). Let \( MK, NL \) be perpendiculars from \( M, N \) so line \( AB \). If \( MK = NL \), show that \( MN \) is parallel to \( AB \).
4. Let $P$ be a point outside a circle $\omega$. Let $PC$ be a tangent to $\omega$, and let a line through $P$ intersect $\omega$ at $A, B$. Prove that $PC^2 = PA \cdot PB$. 

5. (COMC 2000) In $\triangle ABC$ points $D, E, F$ are on sides $BC, CA, AB$, respectively. $\angle AFE = \angle BFD, \angle BDF = \angle CDE, \angle CED = \angle AEF$. Show that $\angle BDF = \angle BAC$. 

6. Prove the perpendicular bisectors in a triangle are concurrent (hint: use property 6 in the Triangles section). 

7. Let $AD, BE$ be the altitudes in $\triangle ABC$. (For simplicity assume $\triangle ABC$ acute, so that $D, E$ lie on sides $BC, AC$; however the properties below will still hold true without this restriction). 
   (a) Prove $\triangle DEC \sim \triangle ABC$. 
   (b) Construct a point $F$ on $AB$ such that $\angle AEF = \angle DEC$. Prove that $\triangle BFD \sim \triangle CDE$. Conclude that $C, H, F$ are collinear. This proves that the altitudes are concurrent. 

8. Let $ABCD$ be a square with side length 1. A point $E$ is constructed outside of this square so that triangle $AEB$ is equilateral. What is the radius of the circle that passes through the points $E, C, D$? (Hint: Consider point $F$ inside the square so that triangle $CFD$ is equilateral.) 

9. Miquel’s Theorem: Let $ABC$ be a triangle. Let $D, E, F$ be points on sides $BC, CA, AB$, respectively. Let $w_1$ be the circle passing through $A, F, E$; $w_2$ be the circle passing through $B, E, D$; $w_3$ be the circle passing through $C, D, E$. Show that these three circles intersect at the same point. (Hint: look at the point of intersection of two of these circles and show it lies on the third circle). 

10. Let $A, B, C, D$ be points on a circle, in this order. If triangle $ABC$ is equilateral ($AB = BC = CA$) show that $BD = AD + CD$. (Hint: Extend $CD$ through $D$ and mark a point $E$ so that $EC = BD$. Prove $BD = ED$.) 

11. (Euclid 2006) Let $AB, BC$ be chords of the circle with $AB < AC$. Let $D$ be the point on the circle such that $AD \perp BC$ and $E$ the point on the circle such that $DE \parallel BC$. Show that $\angle EAC + \angle ABC = 90^\circ$. 

12. (COMC 2007) Let $CBAD$ be a trapezoid with $BA \parallel CD, AB \perp BC$. Assume $BA = 9, BC = 24, CD = 18$, and let $BD, CA$ intersect at $E$. 
   (a) Prove $DE : EB = 2 : 1$. (b) Find the area of $\triangle DEC$. (c) Find the area of $\triangle DAE$. 

13. (COMC 2011) $BDEC$ is a cyclic quadrilateral, inscribed in a circle $\omega$. $BC$ is a diameter of $\omega$. $BD$ and $CE$ intersect at a point $A$. $BC = \sqrt{901}, BD = 1, DA = 16$. What is the length of $EC$? 

14. (COMC 2005) We are given a semicircle with diameter $AB$. A point $P$ is chosen on $AB$, and points $D, E$ on the semi-circle so that $\angle PDO = \angle EDO; \angle DEO = \angle OEB$, and $DP \perp AB$. Find $\angle DOP$. 

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15. Let $ABC$ be a triangle, and $w$ a circle passing through $A, B, C$. Let the angle bisectors of angles $\angle BAC, \angle ACB, \angle ABC$ intersect at point $I$. Let $AI$ intersect the circle $w$ at $M$. Prove that $MB = MI = MC$.

16. (Euclid 2010) Points $A, B, P, Q, C, D$ lie on a line in this order. The semicircle with diameter $AC$ has centre $P$, and the semicircle with diameter $BD$ has centre $Q$. The semicircles intersect at $R$. If $\angle PRQ = 40^\circ$, find $\angle ARD$.

17. (University of Toronto Math Club) Points $P, Q$ are on sides $AB, BC$ of a square $ABCD$ so that $BP = BQ$. Let $S$ be a point on $PC$ so that $BS$ is perpendicular to $PC$. Find $\angle QSD$.

18. Archimedes’ Broken Chord Theorem: Let $A, P, B$ be three points on a circle in this order, so that $AP = PB$. Let $C$ be a point on the circle between $P$ and $B$, so that $C$ and $A$ are on different sides of line $PB$. Let $M$ be a point on $AC$ such that $PM$ is perpendicular to $AC$. Show that $AM = MC + CB$. (Hint: construct point $C'$ on $AC$ so that $C'M = MC$. Now prove that $AC' = CB$.)

19. In a triangle $ABC$, $\angle ABC = 120^\circ$, $\angle BAC = 40^\circ$. The line $AB$ is extended through $B$ to a point $D$ so that $AD = BC + 2AB$. Find $\angle DCA$. (Hint: let $M$ be such that $DM = AB$.)

20. (Euclid 2009) Let $B$ be a point outside a circle $\omega$ with centre $O$ and radius $r$. Let $BA$ be a tangent from $B$ to $\omega$. Let $C$ be a point on the circle, and $D$ be a point inside the circle so that $B, C, D$ lie on a line (in this order). Assume $OD = DC = CB$. Prove that $DB^2 + r^2 = BA^2$.

(Hint: Extend $BD$ through $D$).

21. (APMO 2010) Let $ABC$ be a triangle with $\angle BAC \neq 90^\circ$. Let $O$ be the circumcenter of $\triangle ABC$ and $\omega$ the circumcircle of $\triangle BOC$. $\omega$ intersects line segment $AB$ at $P$ different from $B$, and line segment $AC$ at $Q$ different from $C$. Let $ON$ be the diameter of $\omega$. Prove that $APNQ$ is a parallelogram.

22. (APMO 2005) Let $ABC$ be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let $I$ be the incenter (intersection of angle bisectors), and $H$ the orthocenter (intersection of altitudes) of triangle $ABC$. Prove that $2\angle AHI = 3\angle ABC$.

23. (CMO 2011) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, $X$ the intersection of $AB$ and $CD$, and $Y$ the intersection of $AD$ and $BC$. Let the angle bisector of $\angle AXD$ intersect $AD, BC$ at $E, F$ respectively and let the angle bisector of $\angle AYB$ intersect $AB, CD$ at $G, H$ respectively. Prove that $EGFH$ is a parallelogram.

24. (CMO 2000) Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB, \angle ABD = 2\angle CDB, AB = CB$. Prove $AD = CD$.

25. (IMO SL 1997) A triangle $ABC$ has circumcircle $\omega$. The angle bisectors of $\angle A, \angle B, \angle C$ intersect $\omega$ again at points $K, L, M$ respectively. Let $R$ be a point on side $AB$. A point $P$ is such that $RP$ is parallel to $AK$ and $BP$ is perpendicular to $BL$. A point $Q$ is such that $RQ$ is parallel to $BL$ and $AQ$ is perpendicular to $AK$. Prove that $KP, LQ, MR$ have a point in common.