Geometry

1 Problems

From last time:

1. Let AD, BE be the altitudes in $\triangle ABC$. (For simplicity assume $\triangle ABC$ acute, so that D, E lie on sides BC, AC; however the properties below will still hold true without this restriction).
(a) Prove $\triangle DEC \sim \triangle ABC$.
(b) Construct a point F on AB such that $\angle AEF = \angle DEC$. Prove that $BFEC$ is cyclic, and therefore $CF \perp AB$.
(c) Let H be the intersection of AD, BE. Prove that $\angle BHD = \angle C = \angle BFD$. Conclude that C, H, F are collinear. This proves that the altitudes are concurrent.
2. Let ABC be a triangle, and w a circle passing through A, B, C. Let the angle bisectors of angles $\angle BAC, \angle ACB, \angle ABC$ intersect at point I. Let AI intersect the circle w at M. Prove that $MB = MI = MC$.
3. (Euclid 2010) Points A, B, P, Q, C, D lie on a line in this order. The semicircle with diameter AC has centre P, and the semicircle with diameter BD has centre Q. The semicircles intersect at R. If $\angle PRQ = 40°$, find $\angle ARD$.
4. (University of Toronto Math Club) Points P, Q are on sides AB, BC of a square $ABCD$ so that $BP = BQ$. Let S be a point on PC so that BS is perpendicular to PC. Find $\angle QSD$.
5. *Archimedes' Broken Chord Theorem:* Let A, P, B be three points on a circle in this order, so that $AP = PB$. Let C be a point on the circle between P and B, so that C and A are on different sides of line PB. Let M be a point on AC such that PM is perpendicular to AC. Show that $AM = MC + CB$. (Hint: construct point C' on AC so that $C'M = MC$. Now prove that $AC' = CB$.)
6. In a triangle ABC, $\angle ABC = 120°$, $\angle BAC = 40°$. The line AB is extended through B to a point D so that $AD = BC + 2AB$. Find $\angle DCA$. (Hint: let M be such that $DM = AB$.)
7. (Euclid 2009) Let B be a point outside a circle ω with centre O and radius r. Let BA be a tangent from B to ω. Let C be a point on the circle, and D be a point inside the circle so that B, C, D lie on a line (in this order). Assume $OD = DC = CB$. Prove that $DB^2 + r^2 = BA^2$. (Hint: Extend BD through D).
8. (APMO 2010) Let ABC be a triangle with $\angle BAC \neq 90°$. Let O be the circumcenter of $\triangle ABC$ and ω the circumcircle of $\triangle BOC$. ω intersects line segment AB at P different from B, and line segment AC at Q difference from C. Let ON be the diameter of ω. Prove that $APNQ$ is a parallelogram.
9. (APMO 2005) Let ABC be an acute angled triangle with $\angle BAC = 60°$ and $AB > AC$. Let I be the incenter (intersection of angle bisectors), and H the orthocenter (intersection of altitudes) of triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.
10. (CMO 2011) Let $ABCD$ be a cyclic quadrilateral whose opposite sides are not parallel, X the intersection of AB and CD, and Y the intersection of AD and BC. Let the angle bisector of

$\angle AXD$ intersect AD, BC at E, F respectively and let the angle bisector of $\angle AYB$ intersect AB, CD at G, H respectively. Prove that $EGFH$ is a parallelogram.

11. (CMO 2000) Let $ABCD$ be a convex quadrilateral with $\angle CBD = 2\angle ADB, \angle ABD = 2\angle CDB$, $AB = CB$. Prove $AD = CD$.
12. (IMO SL 1997) A triangle ABC has circumcircle ω. The angle bisectors of $\angle A, \angle B, \angle C$ intersect ω again at points K, L, M respectively. Let R be a point on side AB. A point P is such that RP is parallel to AK and BP is perpendicular to BL. A point Q is such that RQ is parallel to BL and AQ is perpendicular to AK. Prove that KP, LQ, MR have a point in common.

A few more olympiad problems:

1. (CMO 1998) A point O is inside parallelogram $ABCD$ so that $\angle AOB + \angle COD = 180°$. Prove that $\angle ODC = \angle OBC$.
2. (USAMO 1990) An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extension at points P and Q. Prove that the points M, N, P, Q lie on a common circle.
3. (USAMO 1993) Let $ABCD$ be a convex quadrilateral such that $AC \perp BD$ and AC intersects BD at a point E. Prove that the reflections of E across AB, BC, CD, DA lie on a common circle.
4. (CMO 1990) Let $ABCD$ be a cyclic quadrilateral and let X be the intersection of its diagonals. From X we drop perpendiculars XA', XB', XC', XD' to sides AB, BC, CD, DA, respectively. Prove that $A'B' + C'D' = B'C' + A'D'$.
5. (APMO 1993) Let $ABCD$ be a rhombus with $\angle ABC = 60°$. Let l be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let CE and AF intersect at M. Prove that $CA^2 = CM \cdot CE$.

And here is a very cool problem that uses the property that a graph is bipartite (i.e. its vertices can be colored in two colors, so that no two vertices of the same color are connected by an edge) iff it has no odd cycles.

(Russia 2000) In a country with 2000 cities, some cities are connected by roads. It is known that through every city there are at most N non-self-intersecting cycles of odd length. Prove that the country can be divided into $2N + 2$ provinces so that no two cities from the same province are connected by a road.
Hint: remove one edge from every odd cycle in the graph. Color the vertices in the graph in 2 ways - using 2 colors, and using $N + 1$ colors.

2 Some Contest Resources

Pre-olympiad level:

1. Canadian Mathematics Competitions: http://cemc.uwaterloo.ca/contests/past_contests.html
2. Art of Problem Solving Forum: http://www.artofproblemsolving.com/Forum/index.php?
3. Problems Problems Problems: http://cemc.uwaterloo.ca/books.html
4. Arthur Engel, *Problem Solving Strategies*
5. Art of Problem Solving Books: http://www.artofproblemsolving.com/Store/contests.php

Olympiad level

1. *Number Theory* by Naoki Sato:
   http://www.artofproblemsolving.com/Resources/Papers/SatoNT.pdf
2. *Inequalities* by Hoojoo Lee:
   http://www.eleves.ens.fr/home/kortchem/olympiades/Cours/Inegalites/tin2006.pdf
3. Canadian IMO Team training website:
   https://sites.google.com/site/imocanada/
4. *Geometry Unbound* by Kiran Kedlaya:
   http://www-math.mit.edu/~kedlaya/geometryunbound/.
5. Yufei Zhao's olympiad website: http://web.mit.edu/yufeiz/www/olympiad.html; in particular: http://web.mit.edu/yufeiz/www/olympiad/geolemmas.pdf
6. The IMO Compendium: http://www.imomath.com/
7. "From the training of the USA IMO team" book series by Titu Andreescu and Zuming Feng - collections of combinatorics, algebra, number theory, combinatorics problems.

The most important thing is to just do a lot of problems. If you are doing a past math contest, it is very important to time yourself. On the other hand, if you are just working on a problem, it is perfectly fine to spend an hour of maybe even a few hours before looking at a hint or a solution, and you should not rush to read the solution the first time you get stuck on a problem, since then you probably won't learn as much.