Russian-style Problems

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Today we will be doing Russian-style problems related to combinatorics. A lot of these have very non-standard solutions and are rather difficult. The following tricks apply to pretty much all problems. If you feel that you are not getting far on a combinatorics-related problem, it is always good to try these.

- **Induction:** "Induction is awesome and should be used to its full potential" - Jacob Tsimerman, winter camp 2010. Seriously, it is awesome. For hard combinatorics problems, don't expect it to be easy application of induction. Think about how you can reduce your problem to a case for a smaller n in "a powerful way". For example, if you have $n + 1$ vertices in a graph, and you want to use the result on n vertices, take out a vertex with very specific properties and not just any vertex.

- **Extremal Principle:** Define some function f whose domain is "the configuration in your problem" (i.e. configuration of points, sequence of numbers, a graph, etc.) and the range is (usually) the positive integers. Usually it is a rather simple function, i.e. the minimum distance between two points, the largest prime dividing a number, etc. Sometimes you have to be clever and come up with a more complicated function (i.e. the sum of pairwise distances between points). Then consider a configuration for which f achieves its minimum or maximum.

- **Notice Interesting Things:** This idea applies to all olympiad problems, however more so for combinatorics problems. After playing around with the problem for some time you will hopefully come up with *useful properties* of "things" in the problem (e.g. points, edges in a graph, numbers in a sequence, differences between numbers, etc.) You can often tell if the property you found is useful or not. Sometimes it is enough to find *one thing* with a specific property. Sometimes you can remove it and use induction.

- **Reduce the Problem:** After noticing some properties in the problem, you can often reduce or transform the problem to one that is more approachable. **Important**: When you reduce the problem, you are often losing some information from the original problem. It can happen that the results in the reduced problem may not hold because you left out some conditions in the original problem. Check if the result in your reduced problem is even true.

- **Try Small Numbers and See a Pattern:** Self-explanatory.

- **Pigeon-Hole Principle:** Again, usually you have to be clever.

- **Try Lots of Things:** For combinatorics problems there is a lot of freedom and a lot of different approaches you can try. It is possible that you think your current approach is the right one, *you feel stubborn* and keep trying to go further with this approach. If you are using it for 2 hours and having absolutely no new ideas on how to proceed, it is probably a good idea to try another approach. *Keep an open mind*, since many combinatorics problems have unexpected solutions.

Combinatorial Geometry

Some tricks relevant specifically to combinatorial geometry:

- Consider the convex hull made up of the points
- Consider the point with the smallest $x-$ or $y-$ coordinate
- Find the triangle (quadrilateral, pentagon, etc.) with the vertices being the points from your set S, so that the area of the triangle is minimal/maximal
- **Helly's Theorem:** If $X_1, X_2, ..., X_n$ are convex subsets of $\mathbb{R}^k$ so that the intersection of any $k+1$ of them is non-empty, then the intersection of all the sets is non-empty.

1. The two legs of a compass are located at two distinct lattice points in the coordinate plane drawn on an infinite sheet of paper. The distance between the two legs cannot be changed. It is allowed to fix one of the legs, and move the other leg to any other lattice point. Is it possible to switch the positions of the two legs after a finite number of steps?

2. A is a convex set in the plane (so that for any two points in A, the line segment joining the two points lies completely in A). Prove that there exists a point O in A, such that for any points X, X' on the boundary of A, such that O lies on line segment XX',

$$\frac{1}{2} \leq \frac{OX}{OX'} \leq 2$$

3. Find all sets S of finitely many points in the plane, no three of which are collinear and such that for any three points A, B, C in S, there is another point D in S such that A, B, C, D (in some order) are the vertices of a parallelogram.

4. A strip of width w is the set of all points which lie on or between two parallel lines that are a distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

5. There are two circles, each with circumference length 1000 cm. 1000 points are marked on the first circle, and on the other circle - several arcs are marked, so that the sum of the lengths of the arcs is less than 1 cm. Prove that it is possible to lay the first circle on the second so that no marked point lies on a marked arc.

6. Find all positive integers n such that in the coordinate plane, there exists a convex n-sided polygon with all vertices having integer coordinates, and whose side-lengths are odd integers, no two of which are equal.

7. In the plane there are finitely many red and blue lines, no two of which are parallel. For every point of intersection of two lines of the same color, there exists a line of the other color passing through that point. Prove that all the lines are concurrent.

8. A convex polygon is given. Prove that there is at most one way to draw several of its diagonals in such a way, that no two diagonals intersect each other and as a result the polygon is partitioned into acute triangles.

9. There are n lines in the plane, all passing through a point O. For any two lines, there is a third line which bisects one of the pairs of vertical angles formed by the two lines. Prove that the n lines divide the $360°$ angle at O into equal angles.

10. A finite collection of squares has total area 4. Show that they can be arranged to cover a square of side 1.

11. Several identical paper squares of n different colors are lying on a rectangular table, with sides of the squares parallel to the sides of the table. Among any n squares of pairwise distinct colors it is possible to find 2 which can be pinned to the table using one pin. Prove that all squares of a certain color can be pinned to the table using $2n - 2$ pins.

12. There are $N \geq 3$ points in the plane. Among the pairwise distances between these points there are at most n different distances. Prove that $N \leq (n + 1)^2$.

Processes

Some tricks relevant specifically to processes:

- Find an invariant - a quantity that does not change; or a half-invariant - a quantity that does not increase/decrease.
- Use discrete continuity - if a quantity changes by -1, 0, or 1 each time; and it is equal to a and b at two different times, then for any integer between a and b, the quantity will be equal to that integer at some point
- Group things (i.e. moves made, part of the configuration, etc.)

1. A number is written in each of the squares of an $m \times n$ grid. It is allowed to switch the sign of all numbers from one row or one column. Prove that eventually it is possible to get a grid, in which the sums of the numbers in every column and every row are non-negative.

2. Ivan has a 52-card deck. He draws the cards from the deck one by one, without putting them back in the deck. Every time before drawing a card he guesses the suit of the card he will draw. He decides to always guess the suit that occurs most frequently in the remaining deck (if there are several such suits, he chooses any one of them). Prove that he will guess the right suit at least 13 times.

3. Two distinct positive integers a, b are written on the board. The smaller of them is erased and instead of it the number $\frac{ab}{|a-b|}$ is written. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

4. A checker is placed in each of the unit squares of a $n \times n$ square, which is part of an infinite chessboard. A *move* is the process of selecting two checkers located in unit squares sharing a side, and using one of the checkers to jump over the other checker into an empty square adjacent by a side to the square in which the other checker is located. The checker that has been jumped over is removed from the board. After several moves it will be impossible to make a move. Prove that this will happen after at least $\lfloor \frac{n^2}{3} \rfloor$ moves.

5. There are 2000 distinct points, every two of which are connected by a line segment. Danny and Cynthia take turns erasing line segments, so that Danny is allowed to erase only one line segment per turn, and Cynthia is allowed to erase two or three line segments per turn. The person after whose move there is a point not connected to any other points loses. Who will win in this game?

6. A $n \times n$ grid is given, $n-1$ squares of which contain a one, and the rest of the squares contain a zero. It is allowed to select a square, subtract 1 from the number in that square, and add 1 to all numbers in the same row and column as this square. Is it possible to get a grid where all numbers in the squares are equal?

7. An infinite strip of paper is given, divided into unit squares, numbered by the integers from left to right (like a number line). Several stones lie in some of the squares, and a square can have more than 1 stone in it. It is allowed to make the following moves:
(a) Remove a stone from each of the squares n and $n+1$ and place one stone into square $n+2$.
(b) Remove two stones from square n and place one stone into each of the squares $n+1, n-2$.
Prove that eventually it is impossible to make any more moves. Also, prove that the final configuration of the stones is always the same regardless of the order in which the moves were made.

8. In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitiors is a clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

Graphs

Some results relevant specifically to graphs:

- The sum of the degrees of the vertices in a graph is even (a degree is the number of edges exiting the graph).
- If a graph with n vertices has no cycles, but there is a path connecting any two vertices, the graph has $n-1$ edges.
- The set of vertices in a graph can be partitioned into two sets A and B such that no two vertices in A are connected by an edge, and no two vertices in B are connected by an edge iff there are no odd cycles in this graph.

Some useful tricks:

- Take a vertex A and consider the disjoint sets $A_1, A_2, ..., A_n$ so that all vertices in A_1 are connected by an edge to A; all vertices in A_{i+1} are connected by an edge to some vertex in A_i, and not connected by an edge to any of the vertices in $\{A\}, A_1, A_2, ...,$ or A_{i-1}.
- Look at specific parts of the graph satisfying various properties
- Color vertices or edges in the graph
- Define your graph, vertices, and edges in a non-standard way
- Create an algorithm which will produce the desired configuration

1. In a country there are several cities and several roads. Every road connects exactly 2 cities. Out of every city exit at least 3 roads. Prove that there is a cycle, the number of cities in which is not divisible by 3.

2. The inhabitants of a village start getting sick with the flu. One day in the morning some of them ate too much ice cream and got sick; and after that day the only way a healthy person would get sick is if they visited a sick friend. Every person in the village is sick for exactly 1 day, and the next day he is immune to the flu virus - he cannot get sick that day. Despite the pandemic every day a healthy person visits all his sick friends. After the pandemic started, nobody got vaccines. Prove that:
(a) If some people got a vaccine before the first day when the pandemic started, and were immune to the flu on the first day, the pandemic can last forever.
(b) If nobody was immune to the flu on the first day, eventually the pandemic will end.

3. In a certain group of 12 people, among any 9 people, there are 5 who know each other. Show that in this group there are 6 people who know each other.

4. In a country with 2010 cities, there are several two-way roads. Every road connects exactly 2 cities. It is possible to travel from any city to any city along the roads. Furthermore, it is possible to do this even if any one of the roads is closed. Two construction companies A and B are playing a game. On every turn a construction company selects, if possible, one of the roads and enforces one-way traffic on that road (if there is already one-way traffic traffic on the road, this road is not allowed to be selected). A company loses its license if after its move it is impossible to travel from some city to some other city. Company A goes first. Can one of the companies guarantee that the other company will lose its license?

5. In a country with 2010 cities, there are several roads. Every road connects exactly 2 cities. Through every city there are at most n different non-self-intersecting cycles of odd length. Prove that the cities can be divided into $n+2$ groups, so that any two cities from two different groups are not connected by a road.

6. There are 100 representatives from 25 countries seating at a round table, 4 representatives from each country. Prove that it is possible to divide the representatives into 4 groups with one representative from each group so that no two representatives from the same group are sitting side by side at the table.

7. An $m \times n$ rectangular board is given where m, n are odd integers. The board is covered with 2×1 dominoes, so that no two dominoes overlap and only the bottom left square of the board is empty (i.e. not covered by any dominoes). At any point in time it is allowed to slide a domino so that it covers an empty square and still stays on the board, and none of the other dominoes are moved during this process. As a result, a new square becomes empty. Prove that after several such moves it is possible to make any corner square on the board empty.

8. $2n+3$ players participate in a chess tournament. Every two play exactly one game. The schedule is set so that no two games are played at the same time, and each player, after playing game, is free for at least n next (consecutive) games. Prove that one of the players who plays in the first game will also play in the last game.