Projective Geometry

Alexander Remorov
alexanderrem@gmail.com

Harmonic Division

Given four collinear points A, B, C, D, we define their *cross-ratio* as:

$$(A, B; C, D) = \frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}} \tag{1}$$

Note that **the lengths are directed**. When the cross-ratio is equal to -1, we say that $(A, B; C, D)$ is a *harmonic bundle*. A particular case which occurs in many problems is when A, C, B, D are on a line in this order.
Let P be a point not collinear with A, B, C, D; we define the *pencil* $P(A, B, C, D)$ to be made up of 4 lines PA, PB, PC, PD. $P(A, B, C, D)$ is called *harmonic* when $(A, B; C, D)$ is harmonic.

Lemma 1: A pencil $P(A, B, C, D)$ is given. The lines PA, PB, PC, PD intersect a line l at A', B', C', D' respectively. Then $(A', B'; C', D') = (A, B; C, D)$.
Proof: Wolog A, C, B, D are collinear in this order. Using Sine Law in $\triangle CPA, \triangle CPB, \triangle DPA$, $\triangle DPB$, we get (the lengths and angles are directed):

$$\frac{\overrightarrow{CA}}{\overrightarrow{CB}} : \frac{\overrightarrow{DA}}{\overrightarrow{DB}} = \frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)} \tag{2}$$

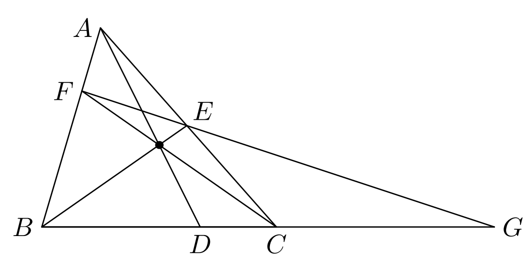
This gives a "trigonometric definition" corresponding to a cross ratio. Lemma 1 follows from (2).

Therefore for any pencil $P(A, B, C, D)$, we can define its cross ratio to be:
$(PA, PB; PC, PD) = (A, B; C, D)$. This definition is ok because of lemma 1.

Corollary 1: **This is extremely useful**. Using the same notation as in lemma 1, if $(A, B; C, D)$ is harmonic then so is $(A', B'; C', D')$.

Lemma 2: In $\triangle ABC$, points D, E, F are on sides BC, CA, AB. Let FE intersect BC at G. Then $(B, C; D, G)$ is harmonic iff AD, BE, CF are concurrent.

Proof: Use Ceva and Menelaus.



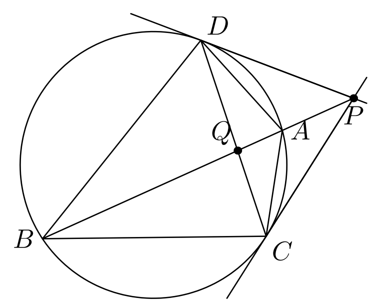
Lemma 3: Consider points A, B, C, D on a circle. Let P be any point on the circle. Then the cross ratio $(PA, PB; PC, PD)$ does not depend on P.
Proof: From (2) it follows that

$$|(PA, PB; PC, PD)| = |\frac{\sin(\angle CPA)}{\sin(\angle CPB)} : \frac{\sin(\angle DPA)}{\sin(\angle DPB)}| = |\frac{CA}{CB} : \frac{DA}{DB}| \quad (3)$$

If $(PA, PB; PC, PD) = -1$, the quadrilateral $ACBD$ is called *harmonic*. From lemma 3, if A, C, B, D are on a circle in this order, and $|\frac{CA}{CB}| = |\frac{DA}{DB}|$, then $ACBD$ is harmonic. The nice thing about lemma 3 is that it allows you to use harmonic pencils for circles.

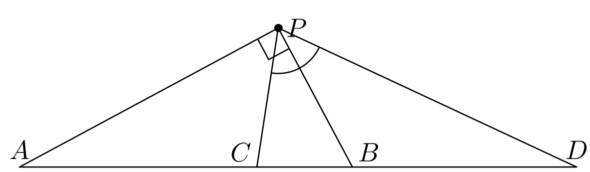
Lemma 4: A point P is outside or on a circle ω. Let PC, PD be tangents to ω, and l be a line through P intersecting ω at A, B (so that P, A, B are collinear in this order). Let AB intersect CD at Q. Then $ACBD$ is a harmonic quadrilateral and $(P, Q; A, B)$ is harmonic.

Proof: $\triangle PAD \sim \triangle PDB \implies \frac{AD}{DB} = \frac{PA}{PD}$. Similarly $\frac{AC}{CB} = \frac{PA}{PC}$. Because $PC = PD$, it follows that $ACBD$ is harmonic.
We can now apply lemma 3. **We take P=C (!)** and consider the intersection of $C(A, B, C, D)$ with line l. Since $ACBD$ is harmonic, the resulting 4 points of intersection form a harmonic bundle, hence $(P, Q; A, B)$ is harmonic.



Corollary 2: Points A, C, B, D lie on a line in this order, and M is the midpoint of CD. Then $(A, B; C, D)$ is harmonic iff $AC \cdot AD = AB \cdot AM$.
Proof: Whenever you see things like $AC \cdot AD$ and circles, trying Power of a Point is a good idea. Assume $AC \cdot AD = AB \cdot AM$. Consider the circle centred at M passing through C, D. Let AT be a tangent from A to this circle. Then $AC \cdot AD = AT^2$. Hence $AB \cdot AM = AT^2$ and $\triangle ATM \sim \triangle ABT$. Since $\angle ATM = 90°$ it follows that $\angle ABT = 90°$. By lemma 4 $(A, B; C, D)$ is harmonic. The converse of the corollary is proved in the same way.

Lemma 5: Points A, C, B, D lie on a line in this order. P is a point not on on this line. Then any two of the following conditions imply the third:
1. $(A, B; C, D)$ is harmonic.
2. PB is the angle bisector of $\angle CPD$.
3. $AP \perp PB$.



Proof: Straightforward application of Sine Law.

Poles and Polars

Given a circle ω with center O and radius r and any point $A \neq O$. Let A' be the point on ray OA such that $OA \cdot OA' = r^2$. The line l through A' perpendicular to OA is called the *polar* of A with respect to ω. A is called the *pole* of l with respect to ω.

Lemma 6: Consider a circle ω and a point P outside it. Let PC and PD be the tangents from P to ω. Then ST is the polar of P with respect to ω.

Proof: Straightforward.

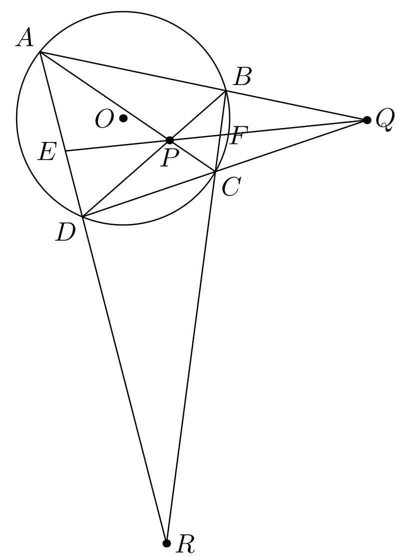
Note: Using the same notation as in lemma 4, it follows that Q lies on the polar of P with respect to ω.

La Hire's Theorem: **This is extremely useful.** A point X lies on the polar of a point Y with respect to a circle ω. Then Y lies on the polar of X with respect to ω.

Proof: Straightforward.

Brokard's Theorem: The points A, B, C, D lie in this order on a circle ω with center O. AC and BD intersect at P, AB and DC intersect at Q, AD and BC intersect at R. Then O is the orthocenter of $\triangle PQR$. Furthermore, QR is the polar of P, PQ is the polar of R, and PR is the polar of Q with respect to ω.

Proof: Let QP intersect BC, AD at F, E, respectively. From lemma 2 it follows that $(R, E; A, D)$ and $(R, F; B, C)$ is harmonic. From lemma 4 it follows that EF is the polar of R. Hence PQ is the polar of R. Similarly PR is the polar of Q and RQ is the polar of P. The fact that O is the orthocenter of $\triangle PQR$ follows from properties of poles and polars.



The configurations in the above lemmas and theorems come up in olympiad problems over and over again. You have to learn to recognize these configurations. **Sometimes you need to complete the diagram by drawing extra lines and sometimes even circles to arrive at a "standard" configuration**.

Problems

Many of the following problems can be done without using projective geometry, however try to use it in your solutions.

0. [**Useful**] M is the midpoint of a line segment AB. Let P_∞ be a point at infinity on line AB. Prove that $(M, P_\infty; A, B)$ is harmonic.

1. [**Useful**] Points A, C, B, D are on a line in this order, so that $(A, B; C, D)$ is harmonic. Let M be the midpoint of AB. Prove that $AM^2 = MC \cdot MD$. (There is a purely algebraic way to do this, as well as a way using poles and polars. Try to find the latter).

2. The tangents to the circumcircle of $\triangle ABC$ at B and C intersect at D. Prove that AD is the symmedian of $\triangle ABC$.

3. AD is the altitude of an acute $\triangle ABC$. Let P be an arbitrary point on AD. BP, CP meet AC, AB at M, N, respectively. MN intersects AD at Q. F is an arbitrary point on side AC. FQ intersects line CN at E. Prove that $\angle FDA = \angle EDA$.

4. Point M lies on diagonal BD of parallelogram $ABCD$. Line AM intersects side CD and line BC at points K and N, respectively. Let C_1 be the circle with center M and radius MA and C_2 be the circumcircle of triangle KCN. C_1, C_2 intersect at P and Q. Prove that MP, MQ are tangent to C_2.

5. (IMO 1985) A circle with centre O passes through vertices A, C of $\triangle ABC$ and intersects its sides BA, BC at distinct points K, N, respectively. The circumcircles of $\triangle ABC$ and $\triangle KBN$ intersect at point B and another point M. Prove that $\angle OMB = 90°$.

6. (Vietnam 2009) Let A, B be two fixed points and C is a variable point such that $\angle ACB = \alpha$, a constant in the range $[0°, 180°]$. The incircle of $\triangle ABC$ with incentre I touches sides AB, BC, CA at points D, E, F, respectively. AI, BI intersect EF at M, N respectively. Prove that the length of MN is constant and the circumcircle of $\triangle DMN$ passes through a fixed point.

7. (Vietnam 2003) Circles C_1 and C_2 are externally tangent at M, and radius of C_2 is greater than radius of C_1. A is a point on C_2 which does not lie on the line joining the centers of the circles. Let B and C be points on C_1 such that AB and AC are tangent to C_1. Lines BM and CM intersect C_2 again at E and F, respectively. Let D be the intersection of the tangent to C_2 at A and line EF. Show that the locus of D as A varies is a straight line.

8. (SL 2004 G8) In a cyclic quadrilateral $ABCD$, let E be the intersection of AD and BC (so that C is between B and E), and F be the intersection of AC and BD. Let M be the midpoint of side CD, and $N \neq M$ be a point on the circumcircle of $\triangle ABM$ such that $\frac{AM}{MB} = \frac{AN}{NB}$. Show that E, F, N are collinear.

9. (SL 2006, G6) Circles ω_1 and ω_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F respectively. Line l is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to l, so that A, E, O_1 are on the same side of l. Prove that AO_1, BO_2, EF and l are concurrent.

Hints

0. This is obvious. However, make sure you know this fact! It is used for several other problems on the handout.

1. It is not hard.

2. It is not hard.

3. There are no circles involved.

4. You almost have a harmonic bundle. Draw the fourth point.

5. Draw tangents from B to the circumcircle of $AKCN$. Now draw another circle with centre B. Complete the diagram.

6. First prove $AN \perp MN$. Do this using the results in the handout.

7. Consider the homothethy carrying one circle to the other.

8. Don't be scared that this is a G8. Complete the diagram.

9. First prove A, D, F are collinear. Complete the diagram.

References

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