1. Consider the dilation carrying $ω$ to the excircle opposite to $A$. Point $E$ is mapped to $F$, which must also be the point of tangency of the excircle to $BC$.

2. Let the excircle $Ω$ be tangent to $BC$ at $F$, and $G$ a point such that $FG$ is diameter of $Ω$. Let $ω$ be the incircle of $△ABC$. Then the homothety with centre $A$ carrying $ω$ to $Ω$ maps $E$ to $G$, so $A,E,G$ are collinear. Hence $E$ is the intersection of $AG$ and $DF$. Therefore $E$ lies on the line connecting the midpoints of $AG,DF$ which is $MI_a$.

3. The dilation with centre $P$ carrying $ω$ to $Γ$ sends $K$ to a point $M$ on arc $AB$ not containing $P$. Line $AB$ is sent to a line $l$ parallel to $AB$ and tangent to $Γ$ at $M$. Angle-chasing finishes the problem.

4. (Proof from Yufei Zhao’s notes on Lemmas in Euclidean Geometry). Extend $KE$ to meet $Γ$ at $M$. $M$ is the midpoint of arc $BC$ (see problem 3) hence $A,I,M$ are collinear. Let $EI$ intersect $ω$ at $F'$. We will show $AF'$ is tangent to $ω$.

Since $∠EF'K,∠MAK$ subtend arcs $EK, MK$ in circles $ω,Γ$ and $MK$ is the image of $KE$ under the homothety carrying $ω$ onto $Γ$ it follows that $∠EF'K = ∠MAK$ so $A,K,I,F'$ are concyclic.

Since $∠BCM = ∠CBM = ∠CKM$ it follows that $△MEC \sim △MCK$ hence $MI^2 = MC^2 = ME \cdot MK$, so $MI$ is tangent to the circumcircle of $△KIM$. Hence $AF'K = ∠AIK = ∠IEK$ so $AF'$ is tangent to $ω$ and $F \equiv F'$.

5. Notice that $∠O_1DO_2 = 90^\circ$. Let $ω_1$ be tangent to $AD, DC$ at $F,E$ and $ω_2$ be tangent to $AD, BD$ at $H,G$. Then $GH, FE$ intersect at $I$. The rest is a simple trig bash.

6. Let $Γ$ be the insimilicentre of $ω$ and $Γ$. By the Monge-d’Alembert Theorem $A', D, T$ are collinear. Hence $A'D, B'E, C'F$ intersect at $T$.

7. $B'C'$ and $BC$ intersect at $N$; they are polars of $A, A'$ respectively. Hence $AA'$ is the polar of $N$. [This is a useful fact!] Similarly $BB'$ is the polar of $M$. Hence $MN$ is the polar of $N$. The result follows.

8. Let $ω$ be tangent to $BC$ at $D$. $AD$ intersect $PQ,ω$ at $K,S$. Considering the dilation carrying the incircle of $△APQ$ to $ω$ it follows that $PK = RQ$ and $MK = MR$. Also $∠RSK = 90^\circ$ hence $MR = MK = MS$ and $MS$ is tangent to $ω$. $AD$ is the polar of $T$ with respect to $ω$ hence $TS$ is tangent to $ω$. The result follows.

9. Let $BI$ intersect $EF$ at $X'$. $EF$ intersect $BC$ at $T$, and $D$ be the point of tangency of $ω$ with $BC$. Then $(T, D; B, C)$ is the angle bisector of $∠FX'D$ hence $X'C \perp BX$. Hence $X \equiv X'$ and $X,Y$ lie on $EF$.

Let $ID$ intersect $EF$ at $N'$. Let $P, Q$ be points on $AB, AC$ so that $N$ lies on $PQ$ and $PQ || BC$. The projections of $I$ onto $AE, ED, FE$ are collinear, so by Simpson’s theorem $I, P, A, Q$ are concurrent.

Since $∠PAI = ∠QAI$ it follows that $IP = IQ$ and $N'P = N'Q$ hence $A, N', M$ are collinear and $N' \equiv N$. So $N$ lies on $ID$.

By angle chasing $I$ is the incentre of $△YXD$ and $△DYX \sim △ABC$. Since $DN$ is the angle bisector of $∠YDX$ (as it contains $I$ it follows that $\frac{NX}{NY} = \frac{DX}{DY} = \frac{AC}{AB}$.

10. Let $U, V, W$ be centers $ω_1, ω_2, ω_3$ respectively. Let $R$ be the intersection of $EF, VW$; $S$ the intersection of $ED, VW$, $T$ the intersection of $FD, UV$. (Some of these might be points of infinity but that’s ok). Then $R, S, T$ are the exsimilicentres between pairs of the three circles. Hence $R$
lies on $BC$, $S$ lies on $AC$, $T$ lies on $AB$ (as they are common external tangents between the pairs of circles). By Monge’s Theorem $R, S, T$ are collinear, hence $\triangle ABC, \triangle DEF$ are perspective with respect to a line. By Desargues’ theorem these triangles are perspective with respect to a point. The result follows.

11. Let $\Gamma, \omega_1(O_1), \omega_2(O_2), \omega_3(O_3), \omega_4(O_4)$ be the circumcircles of the $ABCD, \triangle APB, \triangle BPC, \triangle CPD, \triangle DPA$, respectively ($\omega(O_1)$ means circle $\omega_1$ with centre $O_1$). Let $\omega_1 \cap \omega_3 = P, N$ and $\omega_2 \cap \omega_4 = P, M$. Then $I$, the point of intersection of $O_1O_3$ and $O_2O_4$ lies on the perpendicular bisectors of $PM, PN$, hence is the centre of the circumcircle $\zeta$ of $\triangle PNM$. Let $AD \cap BC = F, AB \cap CD = G$. Then $OE \perp FG$ by Brocard’s Theorem, and it suffices to show $O I \perp FG$ (as then $O, I, E$ are collinear). By the radical axis theorem, $PM, AD, BC$ are concurrent at $F$ and $PN, AB, CD$ are concurrent at $G$. Since $F$ lies on the radical axes of circles $\zeta, \omega_3$ and $\omega_3, \Gamma$ it follows that $F$ lies on the radical axis of $\omega_3, \Gamma$. Similarly $G$ lies on the radical axis of $\omega_3, \Gamma$. So $OE \perp FG$ and the result follows.

12. By Thébault’s theorem $O_1, I, O_2$ are collinear. After some angle chasing we get $I$ is the midpoint of $O_1O_2$. Assume $l$ passes through $M$. Then $\angle O_1MO_2 = 90^\circ$. Also $\angle O_1DO_2 = 90^\circ$ hence $O_1, D, M, O_2$ lie on a circle with centre $I$. Hence $ID = IM$. Let the sides of the triangle be $a, b, c$ and $F$ be the point of tangency of the incircle with $BC$. Then $2BF = BD + DM$ hence $a + c - b = c \cdot \frac{a^2 + c^2 - b^2}{2ac} + \frac{a}{2}$. Simplifying we get $c + b = 2a$. Note: You should not be afraid of using trig bash in your solutions. However, first try to look for a purely geometric solution; use trig bash only when you know where it is going (and not just thoughtless length calculations).

13. Let $\{K\} \equiv CI \cap FE, \{G\} \equiv BI \cap EF$. Then $BK \perp CK$ and $BG \perp CG$. Hence $\{H\} \equiv BK \cap CG$. Let $J$ be the midpoint of $EF$. Let $P'$ be the intersection by $HJ$ and $DM$. It suffices to prove that $P'$ is the midpoint of $DM$.

Let $S$ be the projection of $H$ onto $EF$ and $Y$ the intersection of $HD$ and $EF$. Since $MD \parallel HS$, in order to prove $P$ is the midpoint of $DM$, it suffices to prove the pencil $H(M, J, Y, S)$ is harmonic, i.e. that $(M, Y', J, S)$ is harmonic. Since $MD \parallel JS$, considering the pencil $P_\infty(M, J, Y, S)$ and intersecting it with $HD$ (where $P_\infty$ is the intersection of $MD$ and $HS$) it suffices to prove $(D, Y; I, H)$ is harmonic.

Since $BG, CH$ and $ID$ are altitudes of $\triangle BIC$ it follows that $EI$ is the angle bisector of $\angle KGD$.

Since $\angle HEI = 90^\circ$ it follows that $D, Y; I, H$ is harmonic and the result follows.

14. Let $\Gamma(O)$ be the circle tangent to the lines $AB, BC, AD$ and let $\omega_1, \omega_2, \omega_3$ be the incircles of triangles $APD, BPC$ and $CPD$ respectively.

Since $A$ is the exsimilicenter of $\omega_1$ and $\Gamma$ and $K$ is the insimilicenter of $\omega_1$ and $\omega_3$, by the Monge-D’Alembert theorem, the line $AK$ intersects the line $OI$ at the insimilicenter of $\Gamma$ and $\omega_3$. Similarly, line $BK$ intersects $OI$ at the same insimilicenter $F$ of $\Gamma$ and $\omega_3$. It suffices to prove that $E$ lies on the line $OI$.

By properties of tangents it follows that $AP + CD = PC + AD$ and $BP + CD = BC + PD$ so there exist circles $\omega_5, \omega_6$ inscribed in quadrilaterals $APCD, BCPD$. Let $X$ be the exsimilicentre of $\omega_1, \omega_3$ and $Y$ the exsimilicentre of $\omega_2, \omega_3$. By Monge-D’Alembert theorem applied to circles $\omega_1, \omega_3, \omega_5$ and to circles $\omega_2, \omega_3, \omega_6$ it follows that $A, C, X$ and $B, D, Y$ are collinear. Let $E'$ be the exsimilicentre of $\Gamma$ and $\omega_3$. By the Monge’s theorem applied to $\Gamma, \omega_1, \omega_3$ it follows that $A, X, E'$ are collinear. So $E'$ lies on $AC$ and on $OI$. Similarly $E'$ lies on $BD$ and $OI$. Hence $E' \equiv F$ and $E, O, I$ are collinear.
15. [Proof by Ivan on AOPS] Let $AB, CD$ meet at $X$, $AD, BC$ meet at $Y$, let $k$ meet $AB, DC, AD, BC$ at $P, Q, R, S$ respectively. Using the tangency properties with respect to $k$ we get:

\[ BA + AD = BA + AR - DR = BP - DR = BS - DQ = BC + CQ - DQ = BC + CD \]

Let $k_1, k_2$ meet $AC$ at $J, L$ respectively. Then $AB + JC = BC + AJ$ and $DA + LC = DC + LA$. Adding and using $BA + AD = BC + CD$ we get $JC + LC = AL + AJ$ hence $AL = JC$. Let the excircle of $\triangle ABC$ on the side $AC$ be $k_3$, and the excircle of $\triangle ADC$ on the side $AC$ be $k_4$. Then $k_3, k_4$ meet $AC$ at $L$ and $J$.

Construct the tangent of $k$ which is parallel to $AC$ (and on the same side of $k$ as $AC$). Let that tangent meet $k$ at $Z$. The dilation about $B$ takes $k_3$ to $k$ and $L$ to $Z$. The negative dilation about $D$ takes $k_4$ to $k$ and $J$ to $Z$. Hence $BL$ and $DJ$ meet at $Z$.

Construct the two missing tangents to $k_1$ and $k_2$ which are parallel to $AC$, let the points of tangency be $M$ and $N$ respectively. Similar dilation arguments show that $B, M, L, Z$ are collinear and $D, N, J, Z$ are also collinear.

Since $JM$ and $LN$ are parallel and are diameters of $k_1$ and $k_2$, then they meet at the centre of dilation which takes $k_1$ to $k_2$, which we know is the point $Z$. Hence $Z$ is the intersection of the common external tangents of $k_1, k_2$.

16. Let $\Gamma$ be the circumcircle of $\triangle ABC$. Let $\omega_1$ intersect $\Gamma$ at $B, D$ and $DC$ at $D, E$. Then $\angle XED = 180^\circ - \angle XBD = \angle ACK$ so $XE || AC$. Simple angle chasing gives $\angle AXY = \angle AYX$; let $\angle AXY = \alpha$. Then $\angle YXE = \alpha$, $XY$ is tangent to $\omega_1$ at $X$ so $\angle XKC = \alpha$ and $XYCD$ is cyclic. By the radical axis theorem applied to $\Gamma$ and the circumcircles of $\omega_1$ and $XYCD$ it follows that $AQ, XY, CD$ are concurrent at a point $O$. Since $XE || YC$ and $XY$ is tangent to $\omega_1, \omega_2$ then the homothety with centre $O'$ taking $\omega_1$ to $\omega_2$ takes $X$ to $Y$ and $E$ to $C$, where $O'$ is the exsimilicentre of $\omega_1, \omega_2$. Since $XY \cap EC = O$ it follows that $O$ is the exsimilicentre of $\omega_1, \omega_2$.

Simple angle chasing gives the circumcircle $\zeta$ of $\triangle XYK$ is tangent to $OK$ at $K$. Since $\angle XKP = \angle PXY, \angle YKP = \angle XYP$ it follows that $\angle XKY = \angle XYP + \angle PXY = \angle XYP = \angle AXY$ so $AB$ is tangent to $\zeta$ at $X$. Similarly $AC$ is tangent to $\zeta$ at $Y$. Hence $KA$ is the polar of $O$ with respect to $\zeta$ (since $XY, CD$ are polars of $K, A$ and intersect at $O$). Let $KA$ intersect $\zeta$ at $K, R$ and $XY$ at $S$. Then $(O, S, X, Y)$ is harmonic (proved in previous problems) and if $M$ is the midpoint of $XY$ then $OR^2 = OK^2 = OQ \cdot OA = OX \cdot OY$ (power of a point) $= OS \times OM$ (property of harmonic division).

Consider the inversion with centre $O$ that fixes points $R, K$. The line $AK$ is carried to a circle passing through $R, K, O$ and if this circle intersects $OA, OS$ at $Q', M'$ respectively then $OR^2 = OK^2 = OQ' \cdot OA = OS \cdot OM'$, Hence $Q \equiv Q'$ and $M \equiv M'$ and $OQRMK$ is cyclic. Also $K, P, M$ are collinear (as $M$ lies on the radical axis of $\omega_1, \omega_2$). Hence $\angle QKP = \angle QKM = \angle QOY$. Since $AY^2 = AR \cdot AK = AQ \cdot AO$ it follows the circumcircle of $\triangle QPY$ is tangent to $AC$ and $\angle QKP = \angle QOY = \angle QYA = \angle QXA$ and we are done.