Projective Geometry Solutions

1. Let ω be the circle with centre O and radius MA. By lemma 4 it follows that C lies on the polar of D with respect to ω. Therefore $AM^2 = MC \cdot MD$.

2. Let AD intersect the circumcircle ω of $\triangle ABC$ at A and F, and let E be the intersection of AD with BC. By lemma 4 $ABFC$ is a harmonic quadrilateral. Then $\frac{BF}{FC} = \frac{AB}{AC}$ and $\frac{\sin(\angle BAF)}{\sin(\angle FAC)} = \frac{\sin(\angle ACB)}{\sin(\angle ABC)}$. But $\frac{\sin(\angle ACB)}{\sin(\angle ABC)} = \frac{\sin(\angle MAC)}{\sin(\angle BAM)}$ where M is the midpoint of BC. Hence $\frac{\sin(\angle BAF)}{\sin(\angle FAC)} = \frac{\sin(\angle MAC)}{\sin(\angle BAM)}$ and the result follows.

3. Let EF meet BC at T. Let PQ meet AB at R. By lemma 2 $P(A, N, R, B)$ is harmonic. Intersecting it with line EF, we get $(F, E; Q, T)$ is harmonic. Also $\angle QDT = 90^\circ$ hence by lemma 5 DQ is the angle bisector of $\angle EDF$.

4. Let the diagonals of $ABCD$ intersect at O, and let A' be the reflection of A about M. Then O - midpoint of AC hence $BD||A'C$. Then BD and $A'C$ intersect at a point at infinity S_∞.
$(D, B; O, S_\infty)$ is harmonic (since $\frac{DO}{OB} = \frac{DS_\infty}{S_\infty B}$), hence the pencil $C(D, B, O, S_\infty)$ is harmonic. The intersection of this pencil with line AM gives four points in harmonic division, hence $(A, A'; K, N)$ is harmonic, hence by problem 1, $MA^2 = MK \times MN$. Since $MP = MA$, it follows that $MP^2 = MK \times MN$. The result follows by Power of a Point.

5. We first complete the diagram. Let D be the point of intersection of KN and AC (wolog D, A, C are collinear in this order). Let AN and KC intersect at P, OP intersect BD at M', BO intersect $M'P$ at H.
Denote the circumcircle of $AKNC$ by ω. By lemma 6 DP is the polar of B with respect to ω; let DP intersect ω at X, Y. Then XY is the polar of B hence BX, BY are tangent to ω. By Ω denote the circle with centre B and radius BX.
By Brokard's theorem $\angle OM'D = \angle M'HO = 90^\circ$ so $DM'HO$ is cyclic. By Power of a Point $BD \cdot BM' = BH \cdot BO = BX^2$ (the last equality follows from the fact that $\triangle BXH \sim \triangle BOX$).
Consider the inversion I with respect to Ω. Circles Ω and ω are orthogonal, hence ω is invariant under I, so $I(K) = A, I(N) = C, I(A) = K, I(C) = N$. Furthermore $I(M), I(K), I(N)$ are collinear and also $I(M), I(A), I(C)$ are collinear (since $BMKN$ and $BMAC$ are cyclic quadrilaterals). Therefore $I(M)$ is the intersection of KN and AC which is D. Hence $BM \cdot BD = BX^2$.
Therefore $BD \cdot BM' = BM \cdot BM$ so $M \equiv M'$ and the result follows.
(There is another solution using spiral similarity. If you have not seen it before, see page 6 of Yufei Zhao's handout on cyclic quadrilaterals: `http://web.mit.edu/yufeiz/www/cyclic_quad.pdf`).

6. Let EF intersect AB at G. Since AE, CD, BF are concurrent, it follows that $(G, D; A, B)$ is harmonic. Since $FD \perp AM$, it follows that MA is the angle bisector of $\angle GMD$. Then by lemma 2 $\angle AMB = 90^\circ$. Similarly $\angle ANB = 90^\circ$. (This is also Lemma 8 from a list of lemmas by Yufei Zhao: `http://web.mit.edu/yufeiz/www/geolemmas.pdf`) Therefore A, N, M, B lie on a circle ω with diameter AB. Then $NM = AB\sin(NBM) = AB\sin(90^\circ - \angle IAB - \angle IBA) = AB\sin(\frac{\angle ACB}{2})$.
Let O be the midpoint of AB. Then O is the centre of ω and $\angle NOM = 2 \times \angle NBM = \angle NBM + \angle NAM = \angle NDI + \angle MDI$ (since $ANID, IMBD$ are cyclic) $=\angle NDM$ so the cir-

cumcircle of $\triangle NMD$ always passes through O and the result follows.

7. Let line l be the common tangent to C_1, C_2 at M. A is the pole of line BC with respect to circle C_1, and A lies on line MA, therefore by La Hire's Theorem, pole of MA with respect to circle C_1 lies on line BC. Consider the homothethy h with centre M which transforms C_1 into C_2. Then $h(B) = E, h(C) = F$ and therefore h will take line BC to line EF. **Polar relation does not change through homothethy** (verify this yourself), so h takes the pole of MA with respect to C_1 to the pole of MA with respect to C_2; the pole of MA with respect to C_1 lies on BC so the pole D' of MA with respect to C_2 lies on EF. D' - pole of MA with respect to C_2 and so lies on tangents to C_2 at M and A. But D' also lies on EF and therefore coincides with point D. So D always lies on tangent to C_2 at point M and the result follows.

8. The main idea is that $AMBN$ is a harmonic quadrilateral . We already have the intersections of AC, BD and AD, BC. Let us "complete" the standard picture by drawing G, the intersection of BA and CD. Let the circumcircle of ABM intersect DC again at P. By Power of a Point, $GD \times GC = GA \times GB = GJ \times GM$, hence $(G, P; C, D)$ is harmonic by corollary 2. Let EF intersect CD at P', AB at K and the circumcircle of AMB at N'. (K is between E and N'). Then (G, A, K, B) is harmonic, the pencil $E(G, A, K, B)$ is harmonic hence (G, D, P', C) is harmonic. Therefore $P \equiv P'$. Then $P(G, A, K, B)$ is harmonic. Intersecting this pencil with the circumcircle of ABM; we get a harmonic quadrilateral since P lies on the circumcircle. But the intersections are precisely the points M, A, N', B. Hence $\frac{MA}{AN'} = \frac{MB}{BN'}$. Therefore $N' \equiv N$ and the result follows.

9. Consider the homothethy h with centre F taking ω_2 to ω. (This idea will be explored further during the summer camp lecture in 2 weeks). Then $h(O_2) = O_1$, $h(l)$ is a line parallel to l and tangent to ω at A' where $A' = h(D)$. Then $OA' \perp l$ and $A' \equiv A$. Therefore A, D, F are collinear. Similarly E, D, B collinear. We now have our "standard" cyclic quadrilateral $AEFB$. Let FE meet AB at T and l at S. D lies on the polar of T (by lemma 6) and since $l \perp OD$, and D lies on l, it follows that l is the polar of G (by lemma 6). Then $(T, S; E, F)$ is harmonic by lemma 4, and the pencil $P(T, S, E, F)$ is harmonic. Intersecting it with line O_1O_2 we get a harmonic bundle $(P_\infty, R; O_1, O_2)$ where P_∞ is a point at infinity and R is the midpoint of O_1O_2 hence A, O_1, D and B, O_2, D are collinear. The result follows.