Sequences

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Warm-up Problem 1: Given a positive integer n, consider a sequence of real numbers $a_0, a_1, \dots, a_n$ defined as $a_0 = \frac{1}{2}$ and $a_k = a_{k-1} + \frac{a_{k-1}^2}{n}$ for $1 \leq k \leq n$. Prove that $1 - \frac{1}{n} < a_n < 1$.

Solution: The expression

$$a_k = a_{k-1} + \frac{a_{k-1}^2}{n} \tag{1}$$

seems rather hard to work with so let us think about how we can rewrite it. If we multiply both sides by n and factor out a_{k-1} on the right side, we get:

$$na_k = a_{k-1}(n + a_{k-1}) \Leftrightarrow \frac{a_k}{a_{k-1}} = \frac{n + a_{k-1}}{n} \tag{2}$$

and if we divide (1) by $a_k a_{k-1}$, we get:

$$\frac{1}{a_{k-1}} = \frac{1}{a_k} + \frac{a_{k-1}}{a_k} \cdot \frac{1}{n} \tag{3}$$

Substituting (2) into (3) and rearranging, we have:

$$\frac{1}{a_{k-1}} - \frac{1}{a_k} = \frac{1}{n + a_k} \tag{4}$$

which is much simpler. If we add (4) for $k = 1, 2, \dots, n$ we get:

$$\frac{1}{a_0} - \frac{1}{a_n} = \sum_{k=1}^{n} \frac{1}{n + a_k} \Leftrightarrow \frac{1}{a_n} = 2 - \sum_{k=1}^{n} \frac{1}{n + a_k} \tag{5}$$

Thus now it suffices to show $\frac{n-2}{n-1} < \sum_{k=1}^{n} \frac{1}{n+a_k} < 1$. The second inequality is easy - using (1) and $a_0 > 0$ we have $0 < a_0 < a_1 < \ldots < a_n$, so $n + a_k > n$ for $1 \leq k \leq n$. To prove the first inequality we note that since $a_n < 1$, it follows that $a_k < 1$ for $1 \leq k \leq n$ and thus $\frac{1}{n+a_k} > \frac{1}{n+1}$, so $\sum_{k=1}^{n} \frac{1}{n+a_k} > \frac{n}{n+1} > \frac{n-2}{n-1}$ and we are done.

Comment: Let us try to explain some motivation for (4). Using just (1) and the fact that $a_0 = 0$, it is difficult to get much further than $0 < a_0 < a_1 < \ldots < a_n$ - because of the order 2 terms $\frac{a_k^2}{n}$ present, and it is not clear how to get a bound on them. How can we get rid of the order 2 terms? Divide both sides by $a_k a_{k-1}$ to get (3). We now have an expression for $\frac{1}{a_{k-1}} - \frac{1}{a_k}$ (which gives a telescoping sum), and it is equal to $\frac{a_{k-1}}{a_k} \cdot \frac{1}{n}$. While this is enough to give us $a_n < 1$ (check this yourself), we realize that more is needed to prove $a_n > 1 - \frac{1}{n}$. At this point we make use of (1) again to get (2), and then (4).

Warm-up Problem 2: The sequence $\{a_n\}_{n=1}^{\infty}$ satisfies $a_1 = 1$ and for $n \geq 1$,

$$a_{2n} = a_n + 1;\ a_{2n+1} = \frac{1}{a_{2n}}$$

Prove that every positive rational number occurs in the sequence exactly once.

Solution: Let us start by writing out the first few terms in the sequence. They are $1, 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{5}{3}, \frac{3}{5}$. We notice that starting from the second term, the even-numbered terms are greater than 1, while the odd-numbered ones are less than 1. Proving this rigorously is straightforward with induction.

Now, every positive rational number can be written as $\frac{a}{b}$ where a, b are positive integers with $\gcd(a, b) = 1$. We will use double induction on a, b. The base case is $a = b = 1$ and is clear since $a_1 = 1$. Assume now for positive integers a, b, satisfying $\gcd(a, b)$, we want to show $\frac{a}{b} \in \{a_n\}_{n=1}^{\infty}$ and we know $\frac{c}{d} \in \{a_n\}_{n=1}^{\infty}$ for all positive integers c, d satisfying $c \leq a - 1, d \leq b$ or $c \leq a, d \leq b - 1$. We can assume $a \neq b$ (or else $a = b = 1$ and this case is already done.)

If $a > b$, then $\frac{a}{b} > 1$, so by our first observation, we want to prove that $\frac{a}{b} = a_{2n}$ for some n. But $a_{2n} = a_n + 1$, so we want to prove $a_n = \frac{a-b}{b}$ for some n - which we know is true by the induction assumption.

If $b > a$, then $\frac{a}{b} < 1$, so by our first observation, we want to prove $\frac{a}{b} = a_{2n+1}$ for some n. But $a_{2n+1} = \frac{1}{a_{2n}}$ so we want to prove $\frac{b}{a} = a_{2n}$ for some n - which we already know how to do - since $a_n = \frac{b-a}{a}$ for some n by the induction assumption. The induction step is complete, so every positive rational number occurs in the sequence at least once.

Finally, we need to prove that no two terms in the sequence are the same. Assume we have $a_m = a_n$ for some $m \neq n$ and $m + n$ minimal. It is clear that $m \neq 1$ and $n \neq 1$ (as $a_k \neq 1$ for $k > 1$). Then by our first observation m and n are both even, in which case $a_{\frac{m}{2}} = a_{\frac{n}{2}}$, or m and n are both odd, in which case $a_{m-1} = a_{n-1}$. Thus in both cases we get a contradiction to the fact that $m + n$ is minimal.

Comment: We see the initial observation that for $n \geq 1$, $a_n > 1$ if n is even, and $a_n < 1$ if n is odd, pretty much solved the problem. Playing around with the first few terms in the sequence is always a good idea, especially if they are reasonably nice small numbers. Induction is another thing to keep in mind when faced with these types of problems (although its application is often more tricky than in this problem.)

Problems on sequences come in all kinds of shapes and sizes. Solving these problems usually involves noticing some non-trivial facts about the sequence and then combining them to prove the desired property. Here are a few things you should consider when doing a problem on sequences:

- Prove the sequence is increasing/decreasing/non-increasing/non-decreasing.
- Prove the elements in the sequence are pairwise distinct.
- Show the terms in the infinite sequence take on finitely many values. Then you can find two terms in the sequence that are equal to each other.
- If you have a non-decreasing sequence of integers that is bounded, then it is eventually constant.
- Prove that the sequence is periodic.

- (Usually does not work for non-trivial problems, but still worth a try) Write down the first few terms in the sequence and try to see some kind of pattern. Then maybe use induction to prove this pattern holds.
- Often you are required to prove a certain property about the nth term in the sequence. Using induction on n may do the trick.
- Often a sequence is defined recursively using a certain expression. It is often useful to play around with this expression and possibly rewrite it in another form.
- Construct a new sequence from the given one.
- Consider some term in the sequence satisfying a specific property.
- (If it is possible) Look at the largest or smallest element in the sequence - especially useful if you are trying to show the sequence is constant.

The problems below are approximately arranged in increasing order of difficulty. For sequence problems, where solutions are often short but somewhat "random", different people may have very different opinions about how difficult a particular problem is. So I cannot guarantee that you will find, say, problem 5, easier than problem 20.

Hints are available at the end of the handout. Look at them *only* if you have worked on a problem for over an hour and still think you are stuck.

There are a lot of problems, so it is extremely unlikely that you will be able to finish them all during the session (or before the end of the camp.) I strongly recommend to still try some problems you don't get to today on your own time later this year. The best way to get good at solving sequence problems (considering that heavy machinery is rarely used) is to just do a lot of practice.

Enjoy!

1 Problems

1. A sequence of real numbers $a_0, a_1, \dots$ is defined as follows. a_0 is an arbitrary real number and for $n \geq 0$, $a_{n+1} = \lfloor a_n \rfloor \{a_n\}$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\{x\} = x - \lfloor x \rfloor$). Prove that $a_n = a_{n+2}$ for n sufficiently large.

2. Let $a_1, a_2, \dots$ be a sequence of integers with infinitely many positive terms and infinitely many negative terms. Suppose that for each positive integer n, the numbers $a_1, a_2, \dots, a_n$ leave n different remainders when divided by n. Prove that each integer occurs exactly once in the sequence.

3. The sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are defined as follows. $a_1 = 1, b_1 = 2$, and for $n \geq 1$:
$$a_{n+1} = \frac{1 + a_n + a_n b_n}{b_n}, \; b_{n+1} = \frac{1 + b_n + a_n b_n}{a_n}$$
Prove that $a_{2008} < 5$.

4. A sequence $\{x_n\}_{n=0}^{\infty}$ is defined as follows. $x_0 = a, x_1 = 2$ and
$$x_n = 2x_{n-1}x_{n-2} - x_{n-1} - x_{n-2} + 1 \text{ for } n \geq 2$$
Find all integers a such that $2x_{3n} - 1$ is a perfect square for all $n \geq 1$.

5. Let k be a positive integer greater than or equal to 3. The sequence $\{a_n\}_{n=k}^{\infty}$ satisfies $a_k = 2k$ and for $n > k$:
$$a_n = \begin{cases} a_{n-1} + 1 \text{ if } \gcd(a_{n-1}, n) = 1 \\ 2n \text{ if } \gcd(a_{n-1}, n) > 1 \end{cases}$$
Prove that the sequence $\{a_n - a_{n-1}\}_{n=k+1}^{\infty}$ contains infinitely many primes.

6. Determine if there exist positive integers a, b such that all terms in the sequence $\{x_n\}_{n=1}^{\infty}$ defined by:
$$x_1 = 2010, x_2 = 2011$$
$$x_{n+2} = x_n + x_{n+1} + a\sqrt{x_n x_{n+1} + b}$$
are integers.

7. Assume sequences of real numbers $a_0, a_1, \ldots, a_{2n}$ and $b_0, b_1, \ldots, b_{2n}$ satisfy the following conditions:
(a) For $i = 0, 1, \ldots, 2n-1$, have $a_i + a_{i+1} \geq 0$;
(b) For $j = 0, 1, \ldots, n-1$, have $a_{2j+1} \leq 0$;
(c) For any integers p, q with $0 \leq p \leq q \leq n$, have $\sum_{k=2p}^{2q} b_k \geq 0$.
Prove that $\sum_{i=0}^{2n}(-1)^i a_i b_i \geq 0$ and determine when equality holds.

8. A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is constructed as follows. a_1 is an arbitrary positive integer greater than or equal to 2, and for $n \geq 2$, a_n is the smallest positive integer that is not coprime with a_{n-1} and not equal to $a_1, a_2, \ldots, a_{n-1}$. Prove that $\{a_n\}_{n=1}^{\infty}$ contains every positive integer except 1.

9. A sequence of real numbers $a_1, a_2, \ldots, a_n$ is given. For each i, $1 \leq i \leq n$, define:
$$d_i = \max\{a_i | 1 \leq j \leq i\} - \min\{a_i | i \leq j \leq n\}$$
and let $d = \max\{d_i | 1 \leq i \leq n\}$.
(a) Prove that for any real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$,
$$\max\{|x_i - a_i| | 1 \leq i \leq n\} \geq \frac{d}{2} \tag{6}$$
(b) Show that there are real numbers $x_1 \leq x_2 \leq \ldots \leq x_n$ such that equality holds in (6).

10. Let p be a prime number greater than 3. Show that there exist integers $a_1, a_2, \ldots, a_k$ such that $-\frac{p}{2} < a_1 < a_2 < \ldots < a_k < \frac{p}{2}$ and:
$$\frac{p - a_1}{|a_1|} \cdot \frac{p - a_2}{|a_2|} \cdots \frac{p - a_k}{|a_k|}$$
is a power of 3.

11. A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ is defined as follows. a_1 is an arbitrary positive integer, and for $n \geq 1$, $a_{n+1} = \frac{a_n}{5}$ if a_n is divisible by 5, and $a_{n+1} = \lfloor a_n\sqrt{5} \rfloor$ if a_n is not divisible by 5. Prove that eventually the sequence is strictly increasing.

12. Find all positive integers n greater than 2 for which there exist n positive integers $a_1, a_2, \ldots, a_n$, not all equal, such that $a_1a_2, a_2a_3, \ldots, a_na_1$ is an arithmetic progression (in this order) with non-zero common difference.

13. A sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ contains every positive integer at least once. For every two distinct positive integers m, n, the sequence satisfies:

$$\frac{1}{1998} < \frac{a_n - a_m}{n - m} < 1998$$

Prove that $|a_n - n| < 2000000$ for all positive integers n.

14. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers. Prove that there exist distinct indices i, j such that $a_i + a_j$ does not divide any of the numbers $3a_1, 3a_2, \ldots, 3a_n$.

15. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers such that $\gcd(a_n, a_{n+1}) > a_{n-1}$ for all $n \geq 1$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

16. Show that there exists an infinite bounded sequence $\{a_n\}_{n=1}^{\infty}$ such that for every two distinct positive integers m, n, it satisfies:

$$|a_m - a_n| \geq \frac{1}{m - n}$$

(A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded if there exist real numbers C_1, C_2 such that $C_1 \leq a_n \leq C_2$ for all positive integers n.)

17. Let $x_0, x_1, \ldots, x_{n_0-1}$ be integers, and let $d_1, d_2, \ldots, d_k$ be positive integers with $n_0 = d_1 > d_2 > \ldots d_k$ and $\gcd(d_1, d_2, \ldots, d_k) = 1$. For every integer $n \geq n_0$, define

$$x_n = \lfloor \frac{x_{n-d_1} + x_{n-d_2} + \ldots + x_{n-d_k}}{k} \rfloor$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ is eventually constant.

18. Let n be a positive integer. Given a sequence $\epsilon_1, \ldots, \epsilon_{n-1}$ with $\epsilon_i = 0$ or $\epsilon_i = 1$ for $1 \leq i \leq n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are defined as follows. $a_0 = b_0 = 1, a_1 = b_1 = 7$, and:

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \epsilon_i = 0 \\ 3a_{i-1} + a_i, & \text{if } \epsilon_i = 1 \end{cases}$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \epsilon_{n-i} = 0 \\ 3b_{i-1} + b_i, & \text{if } \epsilon_{n-i} = 1 \end{cases}$$

for $i = 1, 2 \ldots, n-1$. Prove $a_n = b_n$.

19. Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers and s a positive integer such that:

$$a_n = \max\{a_k + a_{n-k} | 1 \leq k \leq n-1\} \text{ for all } n > s$$

Show that there exist positive integers $l \leq s$ and N such that $a_n = a_l + a_{n-l}$ for all $n \geq N$.

20. Suppose there exists a sequence of positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for $k = 2, \ldots, n-1$. Prove that $n \leq 4$.

21. Suppose that every positive integer has been colored either red or blue. Prove that there exists an infinite sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ such that the integers $a_1, \frac{a_1+a_2}{2}, a_2, \frac{a_2+a_3}{2}, a_3, \ldots$ all have the same color.

22. Let $s_1, s_2, s_3, \ldots$ be an infinite sequence of rational numbers, such that not all of its terms are the same. Let $t_1, t_2, t_3, \ldots$ be another infinite sequence of rational numbers, such that not all of its terms are the same. Suppose $(s_i - s_j)(t_i - t_j)$ is an integer for all i, j. Prove that there exists a rational number r, such that $(s_i - s_j)r$ and $\frac{t_i - t_j}{r}$ are integers for all i, j.

23. Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the sub-sequences $s_{s_1}, s_{s_2}, s_{s_3}, \ldots$ and $s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, \ldots$ are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

24. Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For every $k \geq 0$, from the sequence $A_k = (x_1, \ldots, x_n)$ construct a new sequence A_{k+1} as follows.
(i) Partition $\{1, 2, \ldots, n\}$ into two disjoint sets I and J, such that

$$|\sum_{i \in I} x_i - \sum_{j \in J} x_j|$$

obtains the smallest possible value over all possible partitions. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.
(ii) Define $A_{k+1} = (y_1, \ldots, y_n)$ where $y_i = x_i + 1$ if $i \in I$ and $y_i = x_i - 1$ if $i \in J$.
Prove that for some k, the sequence A_k contains an element x such that $|x| \geq \frac{n}{2}$.

2 Problem Sources

Warm-up Problem 1: Finland 1980. Warm-up Problem 2: IberoAmerican MO 2009. Remaining problems:

1. IMO SL 2006.
2. IMO SL 2005.
3. Russia MO 2008.
4. Baltic Way MO 2005.
5. China MO 2010.
6. IberoAmerican MO 2010.
7. China TST 2010.
8. China TST 2010.
9. IMO SL 2007.
10. China Girls MO 2006.
11. Russia MO 2003.
12. Iran TST 2011.
13. Russia MO 1999.
14. IMO SL 2008.
15. IMO SL 2008.
16. Russia MO 1978.
17. USA TST 2011.
18. IMO SL 2009.
19. IMO SL 2010.
20. IMO SL 2009.
21. China TST 2008.
22. USAMO 2009.
23. IMO SL 2009.
24. IMO SL 2007.

Except for warm-up problem 1 and problems 13 and 16, the solutions to the above problems can be found on Mathlinks. If you have trouble finding a solution and want one, feel free to e-mail me.

3 Hints [Spoiler Alert - Look only if you are stuck!]

1. Consider the sequence $\lfloor a_n \rfloor$.
2. Try looking at the first few terms in the sequence.
3. Think about how to combine the expressions for a_{n+1} and b_{n+1} to get something simpler.
4. Construct a new sequence.
5. Play around with what the terms in the sequence may look like.
6. Multiply the expression given in the problem by something.
7. Use induction on n.
8. Play around with what the terms in the sequence may look like; consider primes.
9. Use the definition of d to get a more explicit expression for it.
10. Work modulo 3.
11. Consider two consecutive terms.
12. You can't construct this sequence for all n.
13. Prove that if $m > n$ and $a_m < a_n$ then $m - n < 2000000$.
14. Assume wlog $a_1 < a_2 < \ldots < a_n$.
15. Use induction on n.
16. Consider $\sqrt{2}$.
17. Prove the sequence is periodic.
18. Let w be the "word" $\epsilon_1\epsilon_2 \cdots \epsilon_{n-1}$; $\bar{w}$ be the "word" $\epsilon_{n-1}\epsilon_{n-2} \cdots \epsilon_1$. Define $a_n = (1,7)^w$, $b_n = (1,7)^{\bar{w}}$. You want to show $(1,7)^w = (1,7)^{\bar{w}}$ using induction on n.
19. Consider $\frac{a_k}{k}$.
20. Rewrite the expression given in the problem. What can you say about the parity of the terms?
21. Assume you keep constructing such a sequence for as long as you can, and it is not possible to "construct" the next term.
22. Think about how you can transform the sequences.
23. How could you relate the two arithmetic progressions?
24. What do absolute values remind you of?