Properties of Integers

Theory

In all of the following problems we will be dealing with integers (usually non-negative integers).
We say an integer a is **divisible** by b if $a = bk$ where k is some integer. This is written as $b|a$. The number b is called the **divisor** of a.
An integer p is called **prime** if it has exactly two divisors: 1 and itself. Every integer can be expressed in a unique way as a product of primes: $a = p_1^{k_1} p_2^{k_2} ... p_n^{k_n}$.

General Rules for Divisibility

1. If a, b are positive integers, and a is divisible by b, then $a \geq b$.
2. If a is divisible by b and k is any integer, then ka is divisible by b.
3. If a is divisible by b, and c is divisible by b, then $a + c$ and $a - c$ are divisible by b.
4. If $a = ck$ and $b = dk$, where k is some integer, then if a is divisible by b, then c is divisible by d.
5. If a is divisible by b and c, where b and c are primes, then a is divisible by bc.
6. If a^k is divisible by a prime p, then a is also divisible by p.

Exercises

1. Prove that the only even prime number is 2.
2. Express the following numbers as products of primes: $6, 14, 25, 27, 210, 1000, 1001, 2010$.
3. Show that the product of any 2 consecutive integers is divisible by 2; the product of any 3 consecutive integers is divisible by 6; the product of any 5 consecutive integers is divisible by 120.
4. Let p and q be prime numbers. How many divisors do the following numbers have: pq, pq^2, p^4, p^3q^2?
5. Show that a number is a perfect square only when the number of its divisors is odd.
6. If $a + 1$ is divisible by 3, prove that $10a - 2$ is divisible by 3.
7. If $m + 2$ and $46 - n$ are both divisible by 11, show that $m + n$ is also divisible by 11.
8. If a^2 is divisible by 6, show that a^2 is divisible by 36.
9. Show that if t is a positive integer and t^2 divides t, then $t = 1$.
10. If a, b are positive integers, and a divides b^2 and b^2 divides a, then $a = b^2$.
11. Show that there is no integer value of x such that $4x^2 - 6x + 13 = 0$.
12. Do there exist positive integers so that the product of their digits is 2010?

Problems

1. Show that if n is even, then $n^3 - 4n$ is always divisible by 48.
2. Show that if $a^2 - 9ab + b^2$ is divisible by 11, then $a^2 - b^2$ is also divisible by 11. (Hint: try to use rule 6 for divisibility and remainders).
3. Show that the number $n^4 + 4$ is not prime.
4. Let p, q be primes. How many divisors does the number $p^n q^m$ have? (n, m are non-negative integers).
5. Prove that there are infinitely many prime numbers.
6. Given m, n are positive integers; m divides n^2, n^2 divides m^3, m^3 divides n^4, n^4 divides m^5... Show that $m = n$.
7. P is a polynomial with all coefficients integers, and a, b, c are different positive integers. Show that it is impossible to have $P(a) = b, P(b) = c, P(c) = a$.
8. Do the previous problem, but now if a, b, c are different integers (not necessarily positive).