Properties of Integers

Theory
In all of the following problems we will be dealing with integers (usually non-negative integers). We say an integer \( a \) is divisible by \( b \) if \( a = bk \) where \( k \) is some integer. This is written as \( b | a \). The number \( b \) is called the divisor of \( a \).

An integer \( p \) is called prime if it has exactly two divisors: 1 and itself. Every integer can be expressed in a unique way as a product of primes: \( a = p_1^{k_1}p_2^{k_2}...p_n^{k_n} \).

General Rules for Divisibility
1. If \( a, b \) are positive integers, and \( a \) is divisible by \( b \), then \( a \geq b \).
2. If \( a \) is divisible by \( b \) and \( k \) is any integer, then \( ka \) is divisible by \( b \).
3. If \( a \) is divisible by \( b \), and \( c \) is divisible by \( b \), then \( a + c \) and \( a - c \) are divisible by \( b \).
4. If \( a = ck \) and \( b = dk \), where \( k \) is some integer, then if \( a \) is divisible by \( b \), then \( c \) is divisible by \( d \).
5. If \( a \) is divisible by \( b \) and \( c \), where \( b \) and \( c \) are primes, then \( a \) is divisible by \( bc \).
6. If \( a^k \) is divisible by a prime \( p \), then \( a \) is also divisible by \( p \).

Exercises
1. Prove that the only even prime number is 2.
2. Express the following numbers as products of primes: \( 6, 14, 25, 27, 210, 1000, 1001, 2010 \).
3. Show that the product of any 2 consecutive integers is divisible by 6; the product of any 3 consecutive integers is divisible by 6; the product of any 5 consecutive integers is divisible by 120.
4. Let \( p \) and \( q \) be prime numbers. How many divisors do the following numbers have: \( pq, pq^2, p^3, p^3q^2 \)?
5. Show that a number is a perfect square only when the number of its divisors is odd.
6. If \( a + 1 \) is divisible by 3, prove that \( 10a - 2 \) is divisible by 3.
7. If \( m + 2 \) and \( 46 - n \) are both divisible by 11, show that \( m + n \) is also divisible by 11.
8. If \( a^2 \) is divisible by 6, show that \( a^2 \) is divisible by 36.
9. Show that if \( t \) is a positive integer and \( t^2 \) divides \( t \), then \( t = 1 \).
10. If \( a, b \) are positive integers, and \( a \) divides \( b^2 \) and \( b^2 \) divides \( a \), then \( a = b^2 \).
11. Show that there is no integer value of \( x \) such that \( 4x^2 - 6x + 13 = 0 \).
12. Do there exist positive integers so that the product of their digits is 2010?

Problems
1. Show that if \( n \) is even, then \( n^3 - 4n \) is always divisible by 48.
2. Show that if \( a^2 - 9ab + b^2 \) is divisible by 11, then \( a^2 - b^2 \) is also divisible by 11. (Hint: try to use rule 6 for divisibility and remainders).
3. Show that the number \( n^4 + 4 \) is not prime.
4. Let \( p, q \) be primes. How many divisors does the number \( p^nq^m \) have? \( (n, m \) are non-negative integers).
5. Prove that there are infinitely many prime numbers.
6. Given \( m, n \) are positive integers; \( m \) divides \( n^2 \), \( n^2 \) divides \( m^3 \), \( m^3 \) divides \( n^4 \), \( n^4 \) divides \( m^5 \). Show that \( m = n \).
7. \( P \) is a polynomial with all coefficients integers, and \( a, b, c \) are different positive integers. Show that it is impossible to have \( P(a) = b, P(b) = c, P(c) = a \).
8. Do the previous problem, but now if \( a, b, c \) are different integers (not necessarily positive).