Properties of Integers - Part 2

Theory

We call r the **remainder** when a is divided by a non-zero integer b if $a = bk + r$, where k, r are integers and $0 \leq r \leq b - 1$. This is denoted by $a \equiv r \mod b$.

General Rules for Remainders

1. If $a \equiv r_1 \mod k$, and $b \equiv r_2 \mod k$ then $a + b \equiv r_1 + r_2 \mod k$, and $ab \equiv r_1 r_2 \mod k$.
2. If $a \equiv r \mod k$, then $a^n \equiv r^n \mod k$.
3. An integer n can give one of exactly k remainders when n is divided by k. The remainders are $0, 1, ..., k - 1$. This is useful when solving a problem by cases.
Note, that we can also look at negative remainders, sometimes this is more convenient than positive remainders. For example, $-1 \mod 31 \equiv 30 \mod 31$.
It is sometimes useful to write an integer in its "decimal notation":
$a_1 a_2 ... a_n = 10^{n-1} a_1 + 10^{n-2} a_2 + ... 10 a_{n-1} + a_n$.

Exercises

1. What is the last digit of the following numbers: 2^{100}, $33^{77} + 77^{33}$, $9999^{999^{99^9}}$?
2. Show that the number $n^5 + 4n$ is divisible by 5 for all integers n. (Hint: consider all possible remainders that n can have when divided by 5).
3. Show that the number $n^2 + 1$ is never divisible by 3 for any integer n.
4. Show that if n is odd, then $n^3 - n$ is divisible by 24.
5. Find the last digit of $1^2 + 2^2 + ... + 99^2$.
6. Find the last two digits of $1^3 + 2^3 + ... + 99^3$. (Hint: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$).
7. Show that the number $ab + ba$ (ab denotes number with digits ab) is divisible by 11.
Prove that an integer is divisible:
i. by 2, if its last digit is even; by 2^k if its last k digits make a number divisible by 2^k.
ii. by 3 (or 9) if the sum of its digits is divisible by 3 (or 9).
iii. by 5 if its last digit is 0 or 5; it is divisible by 10 is the last digit is 0.
iv. by 11 is the sum of odd numbered digits and even numbered digits is divisible by 11.
8. For which digits a and b is the number $9a67b$ divisible by 36?
9. For which digits c and d is the number $42c4d$ divisible by 72?
10. Show that if for two integers a and b, if $a^2 + b^2$ is divisible by 3, then a and b are both divisible by 3. (Hint: what remainders can a square of an integer give when divided by 3?)

Problems

1. Find the last 3 digits of the number 7^{9999}. (Hint: find 7^4 first).
2. Show that if n is any positive integer greater than 1, then $3^p + 1$ is not divisible by 2^p. (Hint: look at cases when p is even and p is odd).
3. Fermat's Little Theorem: Let p be a prime number and a is a number not divisible by p. Then $a^{p-1} \equiv 1 \mod p$.
i. Show that if $1 \leq i < j \leq p - 1$, it is impossible to have $ai \equiv aj \mod p$. (Hint: difference).
ii. Using i, show the set $\{a(\mod p), 2a(\mod p), 3a(\mod p), ..., (p-1)a(\mod p)\}$ has $p - 1$ distinct elements and also cannot contain 0.
iii. Conclude that $\{a(\mod p), 2a(\mod p), 3a(\mod p), ..., (p-1)a(\mod p)\}$ is the same as the set $\{1, 2, ..., p - 1\}$.
iv. Show that the number $(p - 1)! = 1 \times 2 \times ... \times (p - 1)$ is not divisible by p.
v. Multiply all elements in iii to get $a \times (2a) \times ... \times (p - 1)a \equiv (p - 1)! \mod p$. Simplify and divide both sides by $(p - 1)!$ to get the result.
4. Every number from 11111 to 99999 inclusive is written on a card. The cards are then arranged in any order in a row. Show that the resulting 444445-digit number cannot be a power of 2. (Hint: it is divisible by 11111).